Final Exam

Math 212 **Multivariable Calculus**

Friday, December 9, 2005: 8:30–11:30am Prof. R. Buckmire

Name: KEY

Directions: Read *all* problems first before answering any of them. There are TEN (10) problems on ELEVEN (11) pages.

This exam is an open-notes, open-book, test. You may **NOT** use a calculator. You must include ALL relevant work to support your answers. Use complete English sentences and CLEARLY indicate your final answer from your "scratch work."

No.	Score	Maximum
1.		**20**
2.		**20**
3.		**20**
4.		**20**
5.		**20**
6		**20**
7.		**20**
8.		**20**
9.		**20**
10.		**20**
TOTAL		**200**

1. *[20 points total.]* **Vector Operations, Equations of Lines.**
Consider the two vectors $\vec{\mathbf{a}} = (1, -2, 1)$ and $\vec{\mathbf{b}} = (-2, 1, 1)$.
(a) (*4 points.*) Compute $\vec{a} \cdot \vec{b}$.

$$\vec{a} \cdot \vec{b} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = -2 - 2 + 1 = -3$$

(b) (*4 points.*) Compute $\vec{a} \times \vec{b}$.

$$\vec{a} \times \vec{b} = \hat{\imath}\left(-2\cdot 1 - 1\cdot 1\right) - \hat{\jmath}\left(1\cdot 1 - (-2)(1)\right) + \hat{k}\left(1\cdot 1 - 2\cdot 2\right) = -3\hat{\imath} - 3\hat{\jmath} - 3\hat{k}$$

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{vmatrix}$$

(c) (*4 points.*) Find the coordinates of the midpoint between **a** and **b**.

$$\text{midpoint} = \left(\frac{1+-2}{2}, \frac{-2+1}{2}, \frac{1+1}{2}\right) = \left(-\frac{1}{2}, -\frac{1}{2}, 1\right)$$

(d) (*4 points.*) Write down the vector equation of the line passing through **a** and **b**.

$$\vec{x}(t) = \vec{a} + (\vec{b} - \vec{a})t$$

$$= \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + \left(\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\right)t = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ 3 \\ 0 \end{pmatrix}t = \begin{pmatrix} 1-3t \\ -2+3t \\ 1 \end{pmatrix}$$

(e) (*4 points.*) Is the point $(-5, 4, 1)$ on the line passing through **a** and **b**? How do you know? **Explain your answer!**

To be on line $\begin{pmatrix} -5 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1-3t \\ -2+3t \\ 1 \end{pmatrix} \Rightarrow$

$-5 = 1 - 3t$
$-6 = -3t$
$2 = t$
and
$4 = -2 + 3t$

If $t = 2$
$-2 + 3(2) = 4 = 4$

Yes, $\begin{pmatrix} -5 \\ 4 \\ 1 \end{pmatrix}$ is on the line!

2. *[20 points total.]* **Equations of Planes, Distance.**

(a) (*5 points.*) Find the (shortest) distance between the planes $2x - y + 3z = 4$ and $2x - y + 3z = 6$. **Explain your answer!**

These planes are parallel to each other since their normal vectors $2\hat{i} - 1\hat{j} + 3\hat{k}$ are equal. Thus the distance between them is the difference in the constant. The distance is 2.

$\boxed{D = 2}$

(b) (*5 points.*) Find the (shortest) distance between the plane $2x - y + 3z = 4$ and the point $(2, 3, 1)$. **Explain your answer!**

$\boxed{D = 0 \quad (2,3,1) \text{ is ALSO on the plane!}}$

$\vec{p} = (2, -1, 3) \quad |\vec{p}| = \sqrt{2^2 + 1^2 + 3^2} = \sqrt{14}$

$\hat{n} = \dfrac{\vec{p}}{|\vec{p}|} = \dfrac{1}{\sqrt{14}}(2, -1, 3)$

Check: $2(2) - 3 + 3(1) \overset{?}{=} 4$, $4 = 4$ ✓

$d = \hat{n} \cdot (\vec{x}_1 - \vec{x}_0)$

$= \dfrac{1}{\sqrt{14}}(2, -1, 3) \cdot (2, 7, 1)$

$= \dfrac{1}{\sqrt{14}}(4 - 7 + 3) = 0$

$\vec{x}_0 = (0, -4, 0)$ is a point on plane

$\vec{x}_1 = (2, 3, 1)$ is the given point to measure

(c) (*10 points.*) Find the (shortest) distance between the planes $2x - y + 3z = 4$ and $2x + y - z = 4$. **Explain your answer!**

If the planes are not parallel, they MUST intersect at some point.

$\boxed{D = 0}$

plane 1: $\vec{n}_1 = (2, -1, 3)$

plane 2: $\vec{n}_2 = (2, 1, -1)$

These planes are not parallel, in fact they are orthogonal!

$\vec{n}_1 \cdot \vec{n}_2 = (2, -1, 3) \cdot (2, 1, -1)$

$= 4 - 1 - 3 = 0$

$2x + y + 3z = 4$

$-(2x + y - z = 4)$

$0 + 2y + 4z = 0$

$y = -2z$

and

$2x = 4 - 5z$

They intersect in a line $\vec{x} = \begin{pmatrix} 2 - \frac{5t}{2} \\ -2t \\ t \end{pmatrix}$

3. *[20 points total.]* **Multivariable Limits and Partial Derivatives.**
Evaluate the following limits.

(b) (*5 points.*) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3 - y^3}{x^3 + y^3}$. **Explain your answer!** $\boxed{\text{DNE}}$

Try along $y = \alpha x$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=\alpha x}} \frac{x^3-(\alpha x)^3}{x^3+(\alpha x)^3} = \lim_{x\to 0} \frac{x^3(1-\alpha^3)}{x^3(1+\alpha^3)} = \frac{1-\alpha^3}{1+\alpha^3}$$

So, if $\alpha = 0$ or 1 one will get different answers
thus, the limit DOES NOT EXIST!

(b) (*5 points.*) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$ **Explain your answer!**

Try $y = \alpha x$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=\alpha x}} \frac{\sin(x^2(1+\alpha^2))}{(1+\alpha^2)x^2} = \lim_{x\to 0} \frac{\sin((1+\alpha^2)x^2)}{(1+\alpha^2)x^2}$$

Use L'Hôpital's Rule

$$\lim_{x\to 0} \frac{2(1+\alpha^2)x\cos((1+\alpha^2)x^2)}{2(1+\alpha^2)x} = \lim_{x\to 0} \cos((1+\alpha^2)x^2) = \boxed{1}$$

Limit exists!

(c) (*10 points.*) Find all the first partial derivatives of $f(x, y, z, t) = x^{y/z}$.

$$\frac{\partial f}{\partial x} = (y/z)\, x^{y/z - 1} = \boxed{\frac{y}{zx}\, x^{y/z}}$$

$f = (x^{1/z})^y$ OR $(x^y)^{1/z}$

$$\frac{\partial f}{\partial y} = (x^{1/z})^y \cdot \ln(x^{1/z}) = \frac{1}{z}(x^y)^{1/z - 1} \ln(x) \cdot x^y$$

$$= x^{y/z}\frac{\ln x}{z} = \frac{1}{z}\ln x \cdot (x^y)^{1/z} = \boxed{\frac{\ln x}{z} \cdot x^{y/z}}$$

$$\frac{\partial f}{\partial z} = (x^y)^{1/z} \cdot \ln(x^y) \cdot \left(-\frac{1}{z^2}\right) = \boxed{-x^{y/z} \frac{y}{z^2} \ln x = f_z}$$

$$\boxed{\frac{\partial f}{\partial t} = 0}$$

4. *[20 points total.]* **Gradient, Directional Derivatives and Tangent Planes.**
Consider the function $f(x, y) = \ln \sqrt{x^2 + y^2}$.
(a) (*5 points.*) Show that $f_{xy} = f_{yx}$ for this function.

$$f = \frac{1}{2} \ln(x^2+y^2)$$

$$f_x = \frac{1}{2} \frac{1}{x^2+y^2} \cdot 2x = \frac{x}{x^2+y^2} \qquad f_{xy} = x \cdot \frac{-1}{(x^2+y^2)^2} \cdot 2y$$

$$f_y = \frac{1}{2} \frac{1}{x^2+y^2} \cdot 2y = \frac{y}{x^2+y^2} \qquad f_{yx} = y \cdot \frac{-1}{(x^2+y^2)^2} \cdot 2x$$

(b) (*5 points.*) Compute $\vec{\nabla} f$ at $(3, 4)$.

$$f_{xy} = f_{yx} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\vec{\nabla} f = (\cancel{\tfrac{1}{2}\ln x^2+y^2}\ f_x, f_y)$$

$$= \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right) \qquad \vec{\nabla} f(3,4) = \left(\frac{3}{25}, \frac{4}{25}\right)$$

(c) (*5 points.*) Compute the directional derivative of $f(x, y)$ in the direction $\vec{v} = -1\hat{\imath} - 2\hat{\jmath}$ at the point $(3, 4)$.

$$\left.\frac{\partial f}{\partial v}\right|_{(x,y)=(3,4)} = \vec{\nabla} f \cdot (-1,-2) = \frac{3}{25} \cdot \left(\frac{-1}{\sqrt{5}}\right) + \frac{4}{25}\left(\frac{-2}{\sqrt{5}}\right)$$

$$\vec{v} = (-1,-2)$$

$$= \frac{-11}{25\sqrt{5}}$$

$$\hat{v} = \frac{-1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}$$

(d) (*5 points.*) Use your answers above to approximate the value of $f(3.01, 3.97)$. (**Do not attempt to evaluate any logarithms**).

$$f(3.01, 3.97) \approx f(3,4) + f_x(3,4)\, 0.01 + f_y(3,4)(-0.03)$$

$$\approx \ln 5 + \frac{3}{25}(0.01) + \frac{4}{25}(-0.03)$$

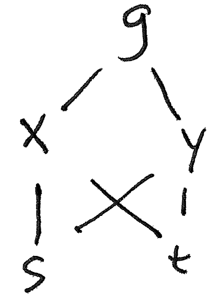
$$\approx \ln 5 - \frac{0.09}{25}$$

So at $(3,4)$ $f_y > f_x$ since f_y makes a larger contribution to the change in f at that point, f is decreasing.

5. *[20 points total.]* **Multivariable Chain Rule.**
Consider two differentiable functions g and f where $g(s,t) = f(x,y)$ and $x = s^2 - t^2$ and $y = t^2 - s^2$. We want to use the Chain Rule to show that

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0.$$

(a) (*5 points.*) Draw the tree diagram for the function g.



$g(x(s,t), y(s,t))$
$= g(s,t)$

(b) (*10 points.*) Write down expressions for $\frac{\partial g}{\partial s}$ and $\frac{\partial g}{\partial t}$.

$$\frac{\partial g}{\partial s} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial g}{\partial s} = g_x 2s + g_y(-2s)$$

$$\frac{\partial g}{\partial t} = g_x \cdot x_t + g_y \cdot y_t$$
$$= g_x \cdot (-2t) + g_y \cdot (2t)$$

(c) (*5 points.*) Show that $t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0$.

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = t \cdot \cancel{g_x 2s} + \cancel{g_y(-2st)} + \cancel{g_x(-2ts)} + \cancel{g_y(2ts)}$$

$$t g_s + s g_t = 0$$

6. *[20 points total.]* **Unconstrained Multivariable Optimization.**
Find the critical points of the $f(x,y) = x^3 + y^3 - 6xy + 1$, and use the second derivative test to classify any local extrem values. What are the *global* extreme values of this function?

$$f_x = 3x^2 - 6y = 0$$
$$f_y = 3y^2 - 6x = 0$$

$$x^2 = 2y \qquad \Rightarrow \qquad \left(\frac{y^2}{2}\right)^2 = 2y$$
$$y^2 = 2x \qquad\qquad y^4 = 8y$$

If $y=0$, $x^2 = 0 \Rightarrow x = 0$ $\qquad y^4 - 8y = 0$

If $y = 2$, $x^2 = 4 \Rightarrow x = \pm 2$ $\qquad y(y^3 - 8) = 0$

But $x = -2$, $y^2 = -4$ (NOT POSSIBLE!) $\qquad y = 0$ or $y^3 = 8 \Leftrightarrow y = 2$

For $x = 2$, $y^2 = 4 \Rightarrow y = \pm 2$

$y = -2$, $x^2 = -4$ NOT POSSIBLE!

Two critical points! $(0,0)$ and $(2,2)$

$$D = f_{xx} f_{yy} - f_{xy}^2$$

$f_{xx} = 6x \qquad f_{xy} = -6$

$f_{yy} = 6y \qquad f_{yx} = -6$

At $(0,0)$

$D(0,0) = (6 \cdot 0)(6 \cdot 0) - (-6)(-6) = -36 \Rightarrow$ SADDLE

$D(2,2) = (6 \cdot 2)(6 \cdot 2) - 6^2 = 108 > 0$ with $f_{xx} > 0$

LOCAL MIN

NO GLOBAL EXTREMES because the ~~bou~~ domain is unbounded.

7. *[20 points total.]* **Constrained Multivariable Optimization, Lagrange Multipliers** Find the points on the ellipse $5x^2 - 6xy + 5y^2 = 4$ which are closest to and furthest from the origin. (**HINT: Optimize the square of the distance between points on the curve and the origin!**)

$g = 5x^2 - 6xy + 5y^2 - 4 \leftarrow$ constraint

$f = x^2 + y^2 \leftarrow$ objective

$$\vec{\nabla} f = \lambda \vec{\nabla} g$$

$$2x = \lambda(10x - 6y)$$

$$2y = \lambda(10y - 6x)$$

$$\left\{\begin{array}{l} x = \lambda(5x - 3y) \\ y = \lambda(5y - 3x) \\ 5x^2 - 6xy + 5y^2 - 4 = 0 \end{array}\right\}$$ Lagrange Multiplier System of Equations

$$\frac{x}{5x-3y} = \frac{y}{5y-3x}$$ Eliminate λ

$$5xy - 3x^2 = 5xy - 3y^2$$

$$x^2 = y^2$$

$x = y$ or $x = -y$

$x = 0$ $y = 0$ is possible but doesn't satisfy CONSTRAINT! $\lambda \neq 0 \Leftarrow (-4 \neq 0)$

If $x = y$

$$5x^2 - 6x^2 + 5x^2 - 4 = 0$$

$$4x^2 - 4 = 0$$

$$x = \pm 1$$

If $x = -y$

$$5x^2 + 6x^2 + 5x^2 - 4 = 0$$

$$16x^2 - 4 = 0$$

$$4x^2 - 1 = 0$$

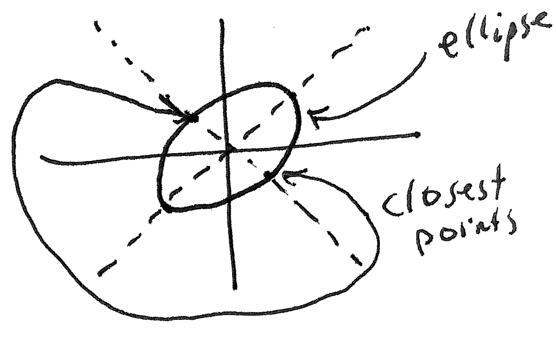
$$x = \pm\frac{1}{2}$$

$(1,1)$ $f(1,1) = 2$

$(-1,-1)$ $f(-1,-1) = 2$

$(\frac{1}{2}, \frac{-1}{2})$ $f(\frac{1}{2}, \frac{-1}{2}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$(\frac{-1}{2}, \frac{+1}{2})$ $f(\frac{-1}{2}, \frac{1}{2}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$



$(1,1)$ and $(-1,-1)$ are FARTHEST.

$(\frac{1}{2}, -\frac{1}{2})$ and $(-\frac{1}{2}, \frac{1}{2})$ are CLOSEST.

8. *[20 points total.]* **Iterated Integration.**

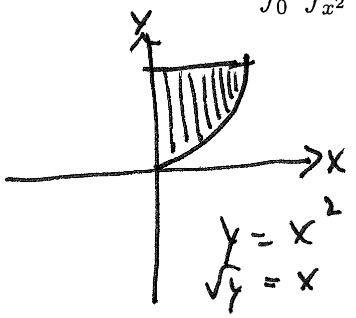
(a) (*6 points.*) $\displaystyle\int_0^{\ln 2}\int_0^{\ln 5} e^{2x-y}\,dx\,dy$

$$\int_0^{\ln 2}\int_0^{\ln 5} e^{2x}e^{-y}\,dx\,dy = \int_0^{\ln 2} 2e^{2x}\Big|_0^{\ln 5} e^{-y}\,dy = \int_0^{\ln 2}(2e^{2\ln 5}-2)e^{-y}\,dy$$

$$2(e^{\ln 25}-1)(-e^{-y})\Big|_0^{\ln 2} = 2\cdot(25-1)\left[-e^{-\ln 2}+1\right] = 2\cdot 24\cdot\left[-\tfrac{1}{2}+1\right]$$

$$= \boxed{24}$$

(b) (*6 points.*) $\displaystyle\int_0^1\int_{x^2}^1 x^3\sin(y^3)\,dy\,dx$



$$= \int_0^1\int_0^{\sqrt{y}} x^3(\sin(y^3))\,dx\,dy$$

$$= \int_0^1 \frac{x^4}{4}\Big|_0^{\sqrt{y}} \sin(y^3)\,dy$$

$$= \frac{1}{4}\int_0^1 ((\sqrt{y})^4 - 0)\sin(y^3)\,dy$$

$$= \frac{1}{4}\int_0^1 y^2\sin(y^3)\,dy$$

$$= \frac{1}{4}\left[-\frac{\cos(y^3)}{3}\right]_0^1 = -\frac{1}{12}\left[\cos 1 - 1\right]$$

$$= \boxed{\frac{1}{12}\left[1-\cos(1)\right]}$$

(c) (*8 points.*) $\displaystyle\int_0^1\int_x^{2x}\int_0^y xyz\,dz\,dy\,dx$

$$\int_0^1\int_x^{2x}\int_0^y xy\frac{z^2}{2}\Big|_0^y\,dy\,dx = \int_0^1\int_x^{2x} x\frac{y^3}{2}\,dy\,dx$$

$$= \int_0^1 x\frac{y^4}{8}\Big|_x^{2x}\,dx = \int_0^1 x\frac{(2x)^4}{8} - x\frac{(x^4)}{8}\,dx$$

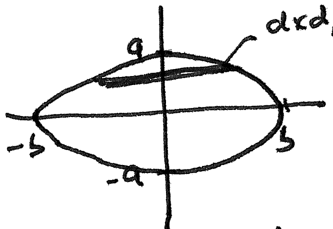
$$= \int_0^1 2x^5 - \frac{x^5}{8}\,dx = \frac{2x^6}{6} - \frac{x^6}{48}\Big|_0^1$$

$$= \frac{1}{3} - \frac{1}{48} = \frac{15}{48} = \boxed{\frac{5}{16}}$$

9. *[20 points total.]* **Green's Theorem.**

By evaluating the line integral $\frac{1}{2}\oint_\Gamma x\,dy - y\,dx$ and applying Green's Theorem, we want to show that the area of an ellipse is πab where $2a$ and $2b$ are the lengths of the minor and major axes of the ellipse and Γ is the closed path in $\mathbb{R}^2$ traced out by the ellipse, which is centered about the origin.

(a) (*6 points.*) Write down a multiple integral which represents the area of the ellipse. (Draw a picture!) **DO NOT EVALUATE THE INTEGRAL!**



$$A = \int_{-a}^{a} \int_{-b\sqrt{1-\frac{y^2}{a^2}}}^{b\sqrt{1-\frac{y^2}{a^2}}} dx\,dy$$

$$\left(\frac{x}{b}\right)^2 + \left(\frac{y}{a}\right)^2 = 1 \Rightarrow x = \pm b\sqrt{1-\frac{y^2}{a^2}}$$

(b) (*10 points.*) Evaluate the line integral $\frac{1}{2}\oint_\Gamma x\,dy - y\,dx$ where Γ is the closed path traced out by the ellipse in the counter clockwise direction.

$$x = b\cos t \qquad t = 0 \to 2\pi$$
$$y = a\sin t$$

$$\int_\Gamma \frac{x}{2}\,dy - \frac{y}{2}\,dx = \int_0^{2\pi} +\frac{b\cos t}{2}\, a\cos t\,dt + \frac{a\sin t\, b\sin t\,dt}{2}$$

$$= \frac{1}{2}\,ab\int_0^{2\pi} \cos^2 t + \sin^2 t\,dt$$

$$= \frac{1}{2}\,ab\int_0^{2\pi} 1\,dt = \frac{1}{2}\,ab\, t\Big|_0^{2\pi} = \frac{1}{2}\cdot ab\cdot 2\pi = \boxed{\pi ab}$$

(c) (*4 points.*) Explain how your above work allows you to find the area of the ellipse to be πab.

By Green's Theorem the value of the line integral is equal to value of the area integral where $\int_\Gamma F\,dx + G\,dy = \iint G_x - F_y\,dx\,dy$

$$\frac{1}{2}\int x\,dy - y\,dx = \iint 1\,dA = \text{Area enclosed by } \Gamma = \pi ab$$

10. *[20 points total.]* **Gradient Fields, Div, Grad and Curl.**
Consider the vector field $\vec{F}(\vec{x}) = F_1(x,y,z)\hat{\imath} + F_2(x,y,z)\hat{\jmath} + F_3(x,y,z)\hat{k}$ where $\vec{x}$ is in $\mathbb{R}^3$. Recall, all gradient fields have zero curl. Recall also, a symmetric matrix is one in which $A_{ij} = A_{ji}$, in other words the transpose of matrix A equals matrix A. For example, $\begin{bmatrix} 1 & 0 & -7 \\ 0 & 2 & 4 \\ -7 & 4 & 3 \end{bmatrix}$ is symmetric.

(a) (*10 points.*) Show that if $\vec{F}$ is a **gradient field**, then the Jacobian of $\vec{F}$ is a **symmetric matrix**.

If $\vec{F}$ is a gradient field, then $\vec{F} = \begin{pmatrix} \phi_x \\ \phi_y \\ \phi_z \end{pmatrix}$

$$F_{\vec{x}} = \begin{pmatrix} \phi_{xx} & \phi_{xy} & \phi_{xz} \\ \phi_{yx} & \phi_{yy} & \phi_{yz} \\ \phi_{zx} & \phi_{zy} & \phi_{zz} \end{pmatrix}$$

Since $\phi_{xy} = \phi_{yx}$, $\phi_{zx} = \phi_{xz}$ and $\phi_{zy} = \phi_{yz}$, $F_{\vec{x}}$ is a symmetric matrix.

(b) (*10 points.*) Show that if the Jacobian of $\vec{F}$ is a **symmetric matrix**, then $\vec{\nabla} \times \vec{F} = \vec{0}$.

~~If $F_{\vec{x}}$ is a symmetric matrix then $\vec{F} = \vec{\nabla}\phi$~~

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \cancel{\hat{\imath}\,\phi_{zy} - 0}$$

$$= \hat{\imath}\left(F_{3_y} - F_{2_z}\right) - \hat{\jmath}\left(-F_{1_z} + F_{3_x}\right) + \hat{k}\left(F_{2_x} - F_{1_y}\right)$$

$$F_{\vec{x}} = \begin{pmatrix} F_{1x} & F_{1y} & F_{1z} \\ F_{2x} & F_{2y} & F_{2z} \\ F_{3x} & F_{3y} & F_{3z} \end{pmatrix}$$

If $F_{1_y} = F_{2_x}$ and $F_{3_x} = F_{1_z}$ and $F_{2_z} = F_{3_y}$ since $F_{\vec{x}}$ is symmetric, so $\vec{\nabla} \times \vec{F} = \vec{0}$.