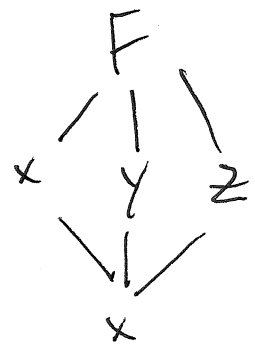
1. (*20 points.*) **Chain Rule, Implicit Function Theorem.**
Consider a surface implicitly-defined as $F(x, y, z) = 0$ which can be written as $z = f(x, y)$ so that $F(x, y, f(x, y)) = 0$.

a. (*10 points*) Use the Chain Rule to show that $\dfrac{\partial F}{\partial x} + \dfrac{\partial F}{\partial z}\dfrac{\partial z}{\partial x} = 0$ and $\dfrac{\partial F}{\partial y} + \dfrac{\partial F}{\partial z}\dfrac{\partial z}{\partial y} = 0$

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$



$$\frac{\partial x}{\partial x} = 1 \qquad \frac{\partial y}{\partial x} = 0$$

$$F_x + F_z \frac{\partial z}{\partial x} = 0$$

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = 0$$

$$F_y + F_z \frac{\partial z}{\partial y} = 0$$

b. (*10 points*) Use the implicit function theorem to obtain the equivalent result,

that is, $\dfrac{\partial z}{\partial x} = -\dfrac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$ and $\dfrac{\partial z}{\partial y} = -\dfrac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$

$$\vec{F}(\vec{x}, \vec{y}) = \vec{0} \qquad \vec{y} = \vec{G}(\vec{x})$$

$$\vec{G}'(\vec{x}) = -\left(\vec{F}_{\vec{y}}\right)^{-1} \vec{F}_{\vec{x}}$$

In this case $F(x,y,z) = \vec{F}(\vec{x}, \vec{y})$ so $\vec{x} = (x, y)$, $\vec{y} = z$

$\vec{F} = F \qquad \vec{G} = f \qquad \vec{G} : \mathbb{R}^2 \to \mathbb{R}^1$, $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^1$

$$\underset{1\times 2}{\vec{G}'(\vec{x})} = \begin{pmatrix} \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{pmatrix} = -\left(\frac{\partial F}{\partial z}\right)^{-1} \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \end{pmatrix} = \begin{pmatrix} -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} & -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} \end{pmatrix}$$

$$\vec{F}_{\vec{y}} = \frac{\partial F}{\partial z}$$

$$\vec{G}'(\vec{x}) = \begin{pmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{pmatrix} =$$

$$\vec{F}_{\vec{x}} = \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \end{pmatrix}$$

2. (*20 points.*) **Partial Differentiation, Gradient Operator.**
Consider $\cos(x+y+z) = xyz$ as an example of $F(x,y,z) = 0$ and $z = f(x,y)$ from Question 1.
a. (*10 points*) Write down F and f, if you can. (If you can't, explain why the requested expression can not be obtained.) How are these two functions different?

$$F(x,y,z) = \cos(x+y+z) - xyz = 0$$

$z = f(x,y)$ is IMPLICITLY DEFINED and can not be written down as an explicit expression

F is explicit,

b. (*10 points*) Write down $\vec{\nabla} F$ and $\vec{\nabla} f$, if you can. (If you can't, explain why the requested expression can not be obtained.) How are these two **vectors** different?

$$\vec{\nabla} F = (F_x, F_y, F_z) \quad \text{in } \mathbb{R}^3$$
$$= (-\sin(x+y+z) - yz, -\sin(x+y+t) - xz, -\sin(x+y+z) - xy)$$

$$\vec{\nabla} f = (f_x, f_y) = \left(-\frac{F_x}{F_z}, -\frac{F_y}{F_z}\right) \quad \text{in } \mathbb{R}^2$$

$$= \left(+\frac{\sin(x+y+z)+yz}{-\sin(x+y+z)-xy}, -\frac{\sin(x+y+z)-xz}{+\sin(x+y+z)-xy}\right)$$

3. *(20 points.)* **Iterated Integration.**

a. *(10 points)* Evaluate $\int_{-3}^{0}\int_{0}^{2}\int_{-1}^{1} \cos(x+y+z) - xyz \, dx\, dz\, dy$

$$\int_{-3}^{0}\int_{0}^{2} \sin(x+y+z) - \frac{x^2}{2}yz \Big|_{-1}^{1} dz\,dy = \int_{-3}^{0}\int_{0}^{2} \sin(1+y+z) - \sin(-1+y+z) - \left(\tfrac{1}{2}yz - \tfrac{1}{2}yz\right) dz\,dy$$

$$= \int_{-3}^{0} -\cos(1+y+z) + \cos(-1+y+z)\Big|_{0}^{2} dy$$

$$= \int_{-3}^{0} -\cos(3+y) + \cos(1+y) + \cos(1+y) - \cos(-1+y)\, dy$$

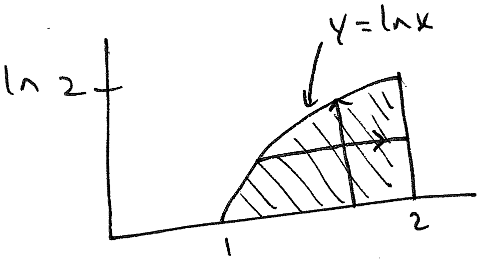
$$= -\sin(3+y) + 2\sin(1+y) + \sin(-1+y)\Big|_{-3}^{0}$$

$$= -\sin 3 + 2\sin 1 - \sin(-1) - \left[-\sin 0 + 2\sin(-2) - \sin(-4)\right]$$

$$= -\sin 3 + 2\sin 1 + \sin 1 - 2\sin(-2) + \sin(-4)$$

$$= \boxed{3\sin 1 + 2\sin 2 - \sin 3 - \sin 4}$$

b. *(10 points)* Evaluate $\int_{1}^{2}\int_{0}^{\ln x} \frac{1}{x}\, dy\, dx$ $= \int_{0}^{\ln 2}\int_{e^y}^{2} \frac{1}{x}\, dx\, dy$



$$= \int_{0}^{\ln 2} \ln x \Big|_{e^y}^{2} dy = \int_{0}^{\ln 2} \ln 2 - \ln(e^y)\, dy$$

$$= \int_{0}^{\ln 2} \ln 2 - y\, dy = (\ln 2)^2 - \frac{(\ln 2)^2}{2}$$

$$= \boxed{\frac{(\ln 2)^2}{2}}$$

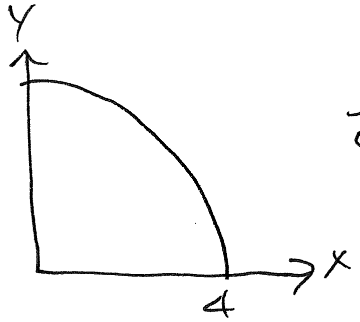
$$\int_{1}^{2}\int_{0}^{\ln x} \frac{1}{x}\, dy\, dx = \int_{1}^{2} \frac{\ln x}{x}\, dx$$

$$= \frac{(\ln x)^2}{2}\Big|_{1}^{2}$$

$$= \frac{(\ln 2)^2}{2} - \frac{(\ln 1)^2}{2} = \frac{(\ln 2)^2}{2}$$

4. (*20 points.*) **Multiple Integration.**

a. (*10 points*) Evaluate $\displaystyle\iint_R ye^x \, dA$ where R is the first quadrant of the circle of radius 4 centered at the origin. (Sketch the region R).



$$\int_0^{\pi/2}\int_0^4 r\sin\theta \, e^{r\cos\theta} \, r\,dr\,d\theta = \int_0^4\int_0^{\pi/2} r^2\sin\theta \, e^{r\cos\theta} \, d\theta$$

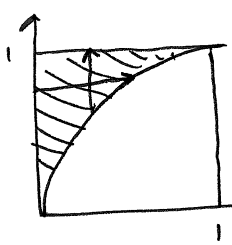
$$= \int_0^4 -re^{r\cos\theta}\Big|_0^{\pi/2} dr = \int_0^4 -re^{r\cdot 0} + re^r \, dr$$

$$= -\frac{r^2}{2}\Big|_0^4 + re^r - e^r\Big|_0^4$$

$$= -8 + 4e^4 - e^4 - (0 - e^0)$$

$$= \boxed{-7 + 3e^4}$$

b. (*10 points*) Consider $\displaystyle\int_0^1\int_{\sqrt{x}}^1\int_0^{1-y} dz\,dy\,dx = \frac{1}{12}$. Re-compute this integral using a different triple integral which represents the same volume.



$$\int_0^1\int_0^{y^2}\int_0^{1-y} dz\,dx\,dy = \int_0^1\int_0^{y^2} 1-y \, dx\,dy$$

$$= \int_0^1 y^2(1-y)\,dy = \int_0^1 -y^3 + y^2 \, dy = -\frac{y^4}{4} + \frac{y^3}{3}\Big|_0^1$$

$$= \frac{1}{3} - \frac{1}{4} = \boxed{\frac{1}{12}}$$

5. (*20 points.*) **Constrained Multivariable Optimization, Lagrange Multipliers**
The "geometric mean" of n numbers is defined as $f(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1 x_2 x_3 \ldots x_n}$. Suppose that $x_1, x_2, \ldots, x_n$ are positive numbers such that $\sum_{i=1}^{n} x_i = x_1 + x_2 + x_3 + \ldots x_n = c$, where c is a constant.

a. (*10 points*) Find the maximum value of the geometric mean of n positive numbers given the constraint that their sum must be equal to a constant. [HINT: Consider f^n instead of f!]

$$\phi = f^n = x_1 x_2 x_3 x_4 \cdots x_n$$

$$g = x_1 + x_2 + x_3 + \cdots + x_n$$

$$\vec{\nabla}\phi = \vec{\nabla}(f^n) = \lambda \vec{\nabla} g$$

$$\nabla(f^n) = (x_2 x_3 x_4 \cdots x_n, x_1 x_3 x_4 \cdots x_n, x_1 x_2 x_4 \cdots x_n, \ldots, x_1 x_2 \cdots x_n)$$

$$\vec{\nabla} g = (1, 1, 1, 1, \ldots, 1)$$

$$x_2 x_3 \cdots x_n = \lambda$$
$$x_1 x_3 \cdots x_n = \lambda$$
$$x_1 x_2 x_4 \cdots x_n = \lambda$$
$$\vdots$$
$$x_1 x_2 \cdots x_{n-1} = \lambda$$
$$x_1 + x_2 + \cdots x_n = c$$

$$\prod_{i=1}^{n} x_i = \lambda x_1$$
$$\prod_{i=1}^{n} x_i = \lambda x_2$$
$$\prod_{i=1}^{n} x_i = \lambda x_3$$
$$\vdots$$
$$\prod_{i=1}^{n} x_i = \lambda x_n$$

~~$\lambda(x_1 + x_2 + \cdots x_n)$~~

$\lambda x_1 = \lambda x_2 = \lambda x_3 = \cdots = \lambda x_n$

either $\lambda = 0$

or

$x = x_1 = x_2 = x_3 = x_n$

If $\lambda = 0$ one $x_i = 0$ (impossible)

$nx = c \Rightarrow x = \frac{c}{n}$

$\phi = \sqrt[n]{\left(\frac{c}{n}\right)^n} = \frac{c}{n} \leftarrow$ MAX

b. (*10 points*) You can deduce from part (a) that the geometric mean of n numbers is always less than or equal to the arithmetic mean, that is:

$$\sqrt[n]{x_1 x_2 x_3 \ldots x_n} \leq \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}$$

Under what conditions will the geometric mean be exactly equal to the arithmetic mean of those same n numbers?

When $x_1 = x_2 = x_3 = \cdots = x_n = X$

geometric mean $= \sqrt[n]{X^n} = X$ ✓

arithmetic mean $= \frac{nX}{n} = X$ ✓

When $\sum_{i=1}^{n} x_i = c$ the max value of $\sqrt[n]{\prod_{i=1}^{n} x_i} = \frac{c}{n}$

so $\sqrt[n]{\prod_{i=1}^{n} x_i} \leq \frac{\sum_{i=1}^{n} x_i}{n}$

EXTRA CREDIT (*10 points.*) **Unconstrained Multivariable Optimization**
Consider $f(x,y) = x^4 + y^4 - 4xy + 1$.
a. (*5 points*) Find the three critical points of $f(x,y)$.

Critical Points occur at $\vec{\nabla} f = \vec{0}$

$f_x = 4x^3 - 4y = 0$ $\qquad y = x^3$

$f_y = 4y^3 - 4x = 0$ $\qquad x = y^3$

$\qquad y = y^9$

$y^9 - y = 0$

$(y^8 - 1)y = 0$

$(y^4 - 1)(y^4 + 1)y = 0$

$(y^2 - 1)(y^2 + 1)(y^4 + 1)y = 0$

$y = 0 \quad y = 1, y = -1$ $\qquad (0,0,1)$

$x = y^3 = 0 \; x = 1, \; x = -1$ $\qquad (1,1,-1)$

$(-1,-1,-1)$

b. (*5 points*) Use the Second Derivative Test to classify each of the three critical points of $f(x,y)$.

$f_{xx} = 12x^2$

$f_{yy} = 12y^2$

$f_{xy} = -4$

At $(0,0,1)$ $\quad D = 0 \cdot 0 - (-4)^2 = -16 < 0 \Rightarrow$ SADDLE

At $(1,1,-1)$ $\quad D = 12 \cdot 12 - (-4)^2 = 144 - 16 = 128 > 0$ LOCAL MIN

$f_{xx} > 0 \quad f_{yy} > 0$

At $(-1,-1,-1)$ $\quad D = 12 \cdot 12 - (-4)^2$

$= 144 - 16 > 0$ $\qquad$ LOCAL MIN

$12 = f_{xx} > 0$ & $f_{yy} > 0$