Math 120 **CALCULUS 2** Spring 2014

Class 27: Friday April 11

Application of Taylor Polynomial Approximations

`Warm-Up`

(a) (WITHOUT Using a calculator) Give an approximate value of $\sqrt{17}$ accurate to zero decimal places. Can you give the value of $\sqrt{17}$ to one decimal place? how about 2 decimal places?

We Can Use Nth Degree Taylor Polynomials To Approximate Functional Behavior Near $x = a$

$$T_N(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \ldots = \sum_{k=0}^{N} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Therefor, if we let $N = 1$, then the first-degree Taylor Polynomial approximation to $f(x)$ at a is

$$T_1(x) = f(a) + f'(a)(x-a)$$

and if we let $N = 2$ the second-degree Taylor Polynomial approximation to $f(x)$ at a is

$$T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

EXAMPLE

Let's look at Taylor Polynomials in the context of $\sqrt{17}$ and the function $f(x) = \sqrt{x}$. Since we know that $\sqrt{16} = 4$ let's look at Taylor Approximations for the square-root function about $a = 16$.

We need to be able to evaluate the first few derivatives of $f(x) = \sqrt{x}$ at $a = 16$

$f(x)$	$f(a)$
$f'(x)$	$f'(a)$
$f''(x)$	$f''(a)$

The first-degree Taylor Polynomial approximation to $f(x) = \sqrt{x}$ at $a = 16$ is

The second-degree Taylor Polynomial approximation $T_2(x)$ to $f(x) = \sqrt{x}$ at $a = 16$ is

An Application of Taylor Polynomials

We can use our Taylor Polynomial approximations to $\sqrt{x}$ to to obtain approximations to $\sqrt{17}$.

$$\begin{aligned}\sqrt{17} \approx T_1(17) &= \\ \sqrt{17} \approx T_2(17) &= \end{aligned}$$

__________ is closer to the actual value of $\sqrt{17}$ than __________.

How could we improve our approximation of $\sqrt{17}$? We could ____________________________________.

Exercise

Find an approximate value of $\sqrt{e}$ to two decimal places.

`DEFINITION: Remainder Term in Taylor Series`

We can write $f(x) = T_N(x) + R_N(x)$ where $T_N(x)$ is the N^{th} degree Taylor Polynomial approximation to $f(x)$ about a and $R_N(x)$ is the remainder term of the Taylor series. This remainder term is the difference between the Taylor Sries approximation to $f(x)$ and the N^{th} degree Taylor Polynomial approximation. The remainder term can be expressed as

$$R_N(x) = \frac{f^{(N+1)}(z)}{(N+1)!}(x-a)^{N+1}$$

where z is some (unknown) number between x and a in the interval of convergence for the Taylor Series of $f(x)$ about a.

EXAMPLE

Let's show that our $T_2(x)$ estimate for $\sqrt{17}$ must be accurate to within 2 decimal places.

Using Taylor Polynomials To Estimate Definite Integrals

We can use Taylor Polynomials to estimate integrands of definite integrals for which we can't use the Fundamental Theorem of Calculus to evaluate exactly.

$\boxed{\text{EXAMPLE}}$

Let's estimate the value of $\displaystyle\int_0^1 \sin\left(x^2\right)\,dx$

$\boxed{\textbf{Exercise}}$

Estimate the value of $\displaystyle\int_0^1 e^{-x^2}\,dx$ to within 0.0001.