Math 120 **CALCULUS 2** Spring 2014

Class 26: Wednesday April 9

Introduction to Taylor Series and Maclaurin Series

`Warm-Up`

(a) What's the equation of a tangent line to the function $f(x) = e^x$ at $x = 0$?

We can Represent ANY Function By A Power Series!

Let's suppose we can represent the function $f(x)$ by a power series centered at a (also known as the power series about a)

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots$$

Let's take the first three derivatives of this function

$$\begin{aligned} f'(x) &= 0 \cdot c_0 + 1 \cdot c_1 + 2 \cdot c_2(x-a) + 3 \cdot c_3(x-a)^2 + 4 \cdot c_4(x-a)^3 + \dots \\ f''(x) &= 0 \cdot c_0 + 0 \cdot c_1 + 2 \cdot c_2 + 3 \cdot 2 \cdot c_3(x-a) + 4 \cdot 3 \cdot c_4(x-a)^2 + \dots \\ f^{(3)}(x) &= 0 \cdot c_0 + 0 \cdot c_1 + 0 \cdot c_2 + 3 \cdot 2 \cdot c_3 + 4 \cdot 3 \cdot 2 \cdot c_4(x-a) + \dots \end{aligned}$$

Look at what happens when we evaluate these derivatives at the value $x = a$,

$$\begin{aligned} f'(a) &= 1 \cdot c_1 \\ f''(a) &= 2 \cdot 1 \cdot c_2 \\ f^{(3)}(a) &= 3 \cdot 2 \cdot 1 \cdot c_3 \end{aligned}$$

By remembering that $f(a) = c_0$ we can get an expression for the first four terms of the power series for $f(x)$ centered about the point $x = a$

$$\begin{aligned} c_0 &= f(a) \\ c_1 &= f'(a) \\ c_2 &= \frac{f''(a)}{2} \\ c_3 &= \frac{f^{(3)}(a)}{3 \cdot 2} \\ c_4 &= \frac{f^{(4)}(a)}{4 \cdot 3 \cdot 2} \\ \vdots &= \vdots \\ c_n &= \frac{f^{(n)}(a)}{n!} \end{aligned}$$

In other words, now that we have an expression for the n^{th} coefficient, we can represent the function $f(x)$ by the following power series:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f''(a)}{3!}(x-a)^3 + \dots = \sum_{k=0}^{\infty} \frac{f(k)(a)}{k!}(x-a)^k$$

This expression is known as the **Taylor Series** (also known as the Taylor Series expansion) for the function $f(x)$ about the point $x = a$. It allows us to find a power series associated with any given function.

`DEFINITION: MacLaurin Series`

The Taylor Series expansion for a given function about the point $a = 0$ is called the **MacLaurin Series** for the function $f(x)$.

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \ldots = \sum_{k=0}^{\infty} \frac{f(k)(0)}{k!}x^k$$

EXAMPLE

Let's show that the Taylor Series expansion for $f(x) = \sin(x)$ about the point $a = 0$ is $\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}$

Let's find the radius of convergence of the Maclaurin Series for $\sin(x)$.

Exercise

Find the MacLaurin Series for $f(x) = e^x$ and show that it converges to e^x for every x-value.

MacLaurin Series That We Should All Know

$$\begin{aligned}
\sin(x) &= \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \\
\cos(x) &= \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \\
e^x &= \sum_{k=0}^{\infty}\frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\
\arctan(x) &= \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots \\
\ln(1+x) &= \sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \\
\frac{1}{1-x} &= \sum_{k=0}^{\infty}x^k = 1 + x + x^2 + x^3 + \dots \\
(a+x)^n &= \sum_{k=0}^{\infty}a^{n-k}x^k\frac{n!}{k!(n-k)!} = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + \dots
\end{aligned}$$

NOTE: The first three of these have infinite radius of convergence, while the other have a radius of convergence of 1. (Their intervals of convergence may vary so you need to check the end points!)

EXAMPLE

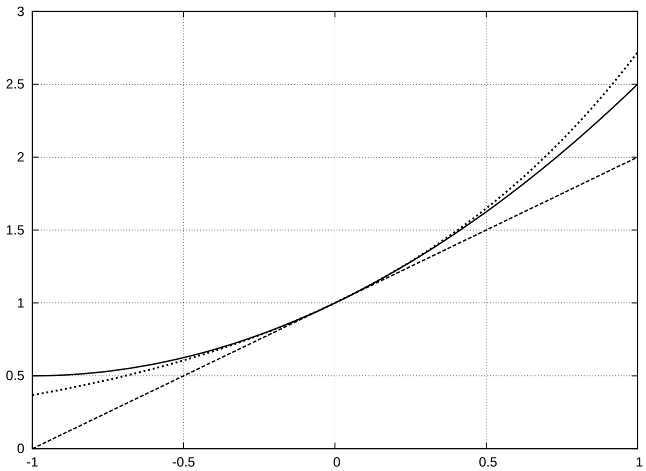
What's the Taylor Series Expansion of $\ln(1-x^2)$ and for what values of x is it valid?

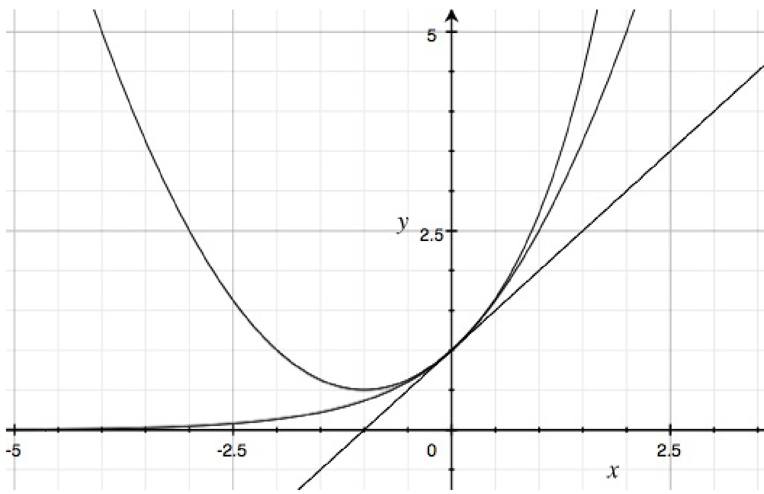
DEFINITION: Taylor Polynomial

The n^{th} degree Taylor Polynomial approximation for a given function $f(x)$ about the point $(a, f(a))$ is the partial sum of the $n+1$ terms of the **Taylor Series** for the function $f(x)$ about the point a.

EXAMPLE

What's the first order Taylor Polynomial approximation to $f(x) = e^x$ at $x = 0$? What's the second-order Taylor Polynomial approximation?





The first order Taylor approximation of a function $f(x)$ at $x = a$ is equivalent to the tangent line approximation to $f(x)$ at a.