Class 25: Friday April 4

Using Power Series To Represent Functions

RECALL

We showed last time that we could represent the function $f(x) = \dfrac{1}{1-x}$ by the power series $\sum_{n=0}^{\infty} x^n$ when $-1 < x < 1$. Can we do this for other functions? Sure!

Exercise

Let's represent the function $\dfrac{1}{1+x^2}$ by a power series. (Find the radius and interval of convergence of this power series.)

EXAMPLE

Remember $\int \dfrac{1}{1+x^2} dx = \arctan(x)$ and $\arctan(1) = \dfrac{\pi}{4}$.

We can use this information to show the amazing result

$$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \ldots = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \qquad \textbf{(Leibniz } \pi \textbf{ Formula)}$$

So we have shown that $\arctan(x)$ can be represented by a power series on the interval $-1 \leq x \leq 1$.

THEOREM

Given a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R > 0$, the function defined by $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ possesses a derivative $f'(x)$ and anti-derivative $F(x)$ on the interval $(a-R, a+R)$ with

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \ldots = \sum_{n=0}^{\infty} c_n n(x-a)^{n-1}$

(ii) $F(x) = \int f(x)dx = C + c_0(x-a) + c_1\dfrac{(x-a)^2}{2} + c_2\dfrac{(x-a)^3}{3} + \ldots = C + \sum_{n=0}^{\infty} c_n \dfrac{(x-a)^{n+1}}{n+1}$

The radius of convergence of $f'(x)$ and $F(x)$ are both R (the same as the radius of convergence of $f(x)$). The intervals of convergence may differ however.

GroupWork

Use this theorem to obtain a power series representation of $\ln(1+x)$. What are the interval of convergence and radius of convergence for the series.

Exercise

Stewart, page 475, #42. Find the sum of the series $\sum_{n=1}^{\infty} \dfrac{4^n}{n5^n}$.