Class 15: Friday February 28

Applications of Integration: Solving Differential Equations

Warm-Up

(a) What mystery function $M(x)$ when differentiated equals $x+1$ and outputs zero when you plug in zero?

(b) Write down a pair of equations that the mystery function $M(x)$ solves.

Introduction To Differential Equations

Differential Equations are equations that contain derivatives. Differential equations are applied widely across many, many fields of science, technology, engineering and mathematics. The reason for this is that we are often interested in investigating how phenomena change over time, and the rate at which this happens. Writing equations to describe this phenomena involves using differential equation.

DEFINITION: **differential equation**

An equation containing the derivative of one or more dependent variables (i.e. y, with respect to one or more independent variables (i.e. x) is said to be a **differential equation**, or DE.

Initial Value Problems

An *initial value problem* consists of a **differential equation** and an **initial condition**. For example,

$$y' = f(x), \qquad y(a) = b$$

The solution of an initial value problem is a specific FUNCTION $y(x)$ about which we know its DERIVATIVE (i.e. $y' = f(x)$) and one value it goes through (i.e. when $x = a, y = b$).

Fundamental Theorem Of Calculus, part 3

The solution of the IVP $y' = f(x), \quad y(a) = b$ is

$$y(x) = b + \int_a^x f(t)dt$$

EXAMPLE

Let's confirm that the proposed solution in FTC part 3 actually is a solution of the given IVP $y' = f(x), \quad y(a) = b$.

Separation of Variables
We can also solve differential equations of the form

$$y' = f(x)g(y), \quad y(a) = b$$

by using a technique called Separation of Variables.
We can re-write y', the derivative of $y(x)$, as $\frac{dy}{dx}$

$$\begin{aligned} \frac{dy}{dx} &= f(x)g(y) \\ \frac{dy}{g(y)} &= f(x)dx \\ \int \frac{dy}{g(y)} &= \int f(x)dx \end{aligned}$$

EXAMPLE
1. If the rate of increase of a population $P(t)$ is proportional to the current population $P(t)$, then a differential equation describing the rate of growth of a population that starts with A members is:

$$\frac{dP}{dt} = kP, \qquad P(0) = A$$

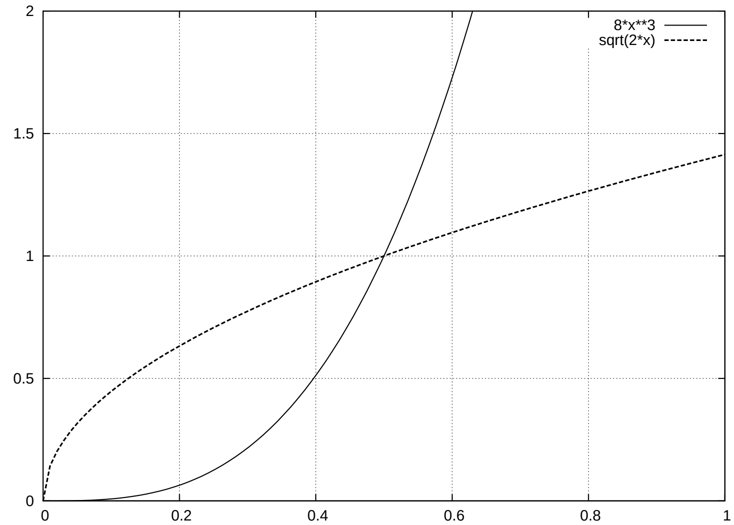
GroupWork

2. Use Separation of Variables to solve the following initial value problem

$$\frac{dy}{dt} = yt, \qquad y(0) = 2$$

3. Check that $y(t) = \dfrac{1}{1-t}$ is the solution to $y' = y^2, y(0) = 1$

A Different Way Of Looking At The Same Shape



`RIGHT-LEFT FORMULA`

We can think of this shape as being bounded by two curves $x = L(y)$ and $x = R(y$ and the lines $y = c$ and $y = d$. In that case, the area A would be given by

$$A = \int_c^d [R(y) - L(y)] \, dy$$

Exercise

What are the functions $x = L(y)$, $x = R(y)$ and the lines $y = c$ and $y = d$ for the area above?

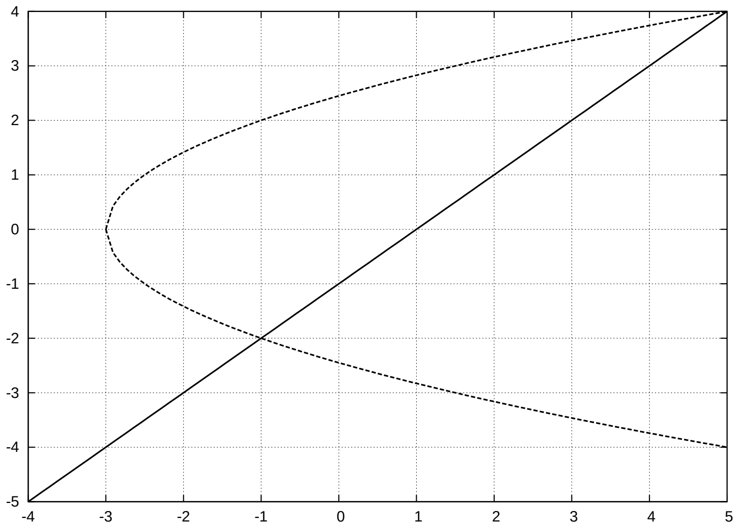
EXAMPLE

Compute the value of A again, this time using horizontal boxes.

Depending on the shape of particular area, you should choose horizontal boxes (i.e. a Right-Left dy integral) or vertical boxes (a Top-Bottom dx integral).

GroupWork

1. Find the area between the line $y = x - 1$ and the parabola $y^2 = 2x + 6$



2. **Stewart, page 369, #7.** Find the area between the curves $y = (x - 2)^2$ and $y = x$.