Math 120 | **CALCULUS 2** | Spring 2014

Class 12: Friday February 21

Improper Integrals, Part 1

`Warm-Up`

(a) Can the area bounded by a finite-valued graph, the y-axis and an infinite interval of the x-axis be finite? TRUE or FALSE?

(b) Suppose p is a known fixed number, evaluate $\lim_{b\to\infty} b^p$

Improper Integrals of the Second Kind

EXAMPLE

1. (a) Sketch a graph of the function $f(x) = 1/x^2$ for $x \geq 0$

(b) Find the area under this curve, from $x = 1$ to $x = b$, for each of the following values of b:
(i) $b = 10$. Area =

(ii) $b = 100$. Area =

(iii) $b = 1000$. Area =

(c) What do you *think* the TOTAL area under the curve to the right of $x = 1$ is?

(d) Let's try to *prove* what this total area is. First find the area under the curve $f(x) = \dfrac{1}{x^2}$ from 1 to b in terms of b.

(e) The total area under the curve to the right of $x = 1$ can be found by plugging in larger and larger values of b to find the number your answers are approaching.

Translating the above sentence into mathematical language, we say we are taking the __________ as b goes to __________ of the area from 1 to b. Do this in the space below...

(f) The mathematical "shorthand" notation for what we did above is:

$$\int_1^\infty \frac{1}{x^2}\,dx \overset{def}{=} \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx$$

The integral on the left hand side is called an **improper integral**. In general, when the limit exists, we say **the improper integral converges**. Otherwise we say **the improper integral diverges**.

Exercise

2. (a) Sketch a graph of the function $f(x) = 1/x$.

(b) Find the area under this curve, to the right of $x = 2$, by evaluating the improper integral $\int_2^\infty \frac{1}{x}\,dx$.

Hint: do this in two steps:

Step 1. Evaluate $\int_2^b \frac{1}{x}\,dx$.

Step 2. Now find the appropriate limit.

Step 3. Does the integral $\int_2^\infty \frac{1}{x}\,dx$ CONVERGE or DIVERGE? (Choose one and explain your answer)

Improper Integrals of the Second Kind

EXAMPLE

3. (a) Previously we found that the total area under $f(x) = 1/x^2$ to the right of $x = 1$ is 1. Now let's find the area under the same curve, but between the y-axis and $x = 1$. First shade in this area in the same graph you sketched above in problem 1.

(b) $\int_0^1 \frac{1}{x^2}\,dx =$

(c) As you can see, this definite integral SEEMS to be undefined, because the integrand $1/x^2$ is undefined at $x = 0$. So this kind of integral is also called an **improper integral**. Let's try a different way to evaluate it, then. Find the area from a to 1, in terms of a (where a is any number between 0 and 1).

(d) What is the area for each of the following values of a?

$a = 0.1$

$a = 0.01$

$a = 0.001$

(e) Considering the results above, what do you *think* the area from 0 to 1 is?

(f) PROVE your answer mathematically.

GROUPWORK

4. Evaluate each of the following improper integrals. (Remember, it's easier if you do each problem in two steps: first evaluate a definite integral "using a or b"; then take the appropriate limit.)

(a) $\int_2^\infty \frac{1}{\sqrt{x}}\,dx =$

(b) $\int_0^2 \frac{1}{\sqrt{x}}\,dx =$

(c) $\int_{-\infty}^\infty e^x\,dx =$

(Hint: break this into two integrals: one from $-\infty$ to 0, the other from 0 to ∞.)