Math 120 | **CALCULUS 2** | Spring 2014

Class 2: Friday January 24

The Definite Integral

`Warm-Up`

Our goal today is to make that eveyone fully understands the connections between Riemann Sums and Definite Integrals. To that end, take 3 minutes and complete the following sentences.

1. A **Riemann Sum** can be written mathematically as ...

2. A **Definite Integral** can be computed from a **Riemann Sum** by ...

AREA as a Riemann Sum

Let's look at a simple area problem for which we know we can find the answer *exactly*.

3. What is the area under the curve $f(x) = 3x$ from the origin ($x = 0$) to some point $x = L$? (HINT: what is the shape that area forms?)

4. Sketch a picture of this area in the figure below. Then our goal will be to compute the exact answer using our knowledge of Riemann Sums and how the connect to definite integrals.

We will use a Riemann Sum using an **equipartition** on the interval [0,L] with N subintervals to approximate this area.

In order to do this you will need to answer a few questions:

5. What is Δx for your partition? (NOTE: An equipartition is what we call the set of points you get when you split up an interval into a number of equal parts.)

6. In general, to find Δx on an equipartition of n subintervals on $[a, b]$ the formula is...

7. What is the formula for the sampling point x_k? (Assume we are doing a RIGHT-HAND Riemann Sum)

8. Now write down a **Right-hand Riemann Sum** to approximate the shaded area.

9. What is a formula for the **exact area** under the curve, using the Riemann Sum? (How do estimates made using Riemann Sums become more accurate?)

We can also represent the exact value of the area by the symbol $\int_0^L 3x\,dx$.

This is called a **definite integral**.

Definition of the Definite Integral

Given a function $f(x)$ defined on an interval $[a, b]$ the definite integral can be defined as

$$\int_a^b f(x)\,dx = \lim_{max \Delta x_k \to 0} \sum_{k=1}^{N} f(x_k)\Delta x_k$$

IF the above limit exists THEN we say that the given function $f(x)$ is **integrable** on the interval $[a, b]$.

THEOREM

If f is integrable on $[a, b]$ and $\Delta x = \dfrac{b-a}{N}$,

$$\int_a^b f(x)\,dx = \lim_{N \to \infty} \sum_{k=1}^{N} f(x_k)\Delta x$$

This is the method we have been using to use Riemann Sums to estimate definite integrals where x_k is some sample point in each subinterval.

Riemann Sum Sampling Point Formulas

Left-Hand Sum	**Right-Hand Sum**	**Midpoint Sum**
$x_k = a + (k-1)\Delta x$	$x_k = a + (k)\Delta x$	$x_k = a + (k)\Delta x$

PAIR WORK

Write down a sentence translating the mathematical symbols $\int_a^b f(x)\ dx$ into English and then compare your sentence with your neighbor's.

Frequently Asked Questions About the Definite Integral

Q: What is the meaning of the $\int$ symbol? The symbol was introduced by Gottfried Leibniz, one of the co-inventors of the Calculus (along with Sir Isaac Newton), and is called an integral sign. It represents the fact that the integral is the limit of sums.

Q: Why is the dx there? The integral is meaningless without the dx to explain what is the interval (and in one what variables) the integration process is occurring. An integral sign $\int$ without a corresponding dx I lke to call an **illegal integral**.

Q: Can we integrate every function? No! Notice that the integral was defined conditionally, if a particular limit exists. However, there is a theorem (on p. 270 of Stewart) that says if f is continuous on $[a, b]$ (or if it has a **finite** number of jump discontinuities) then f is integrable on $[a, b]$. Most of the functions we will be seeing in math 120 will be integrable.

Q: What IS the definite integral, really? Remember, in the end, the definite integral is just a (very fancy) way of writing a number. It can be *interpreted* to have significance in various physical contexts. The primary context is as a measure of "signed area" under a curve.

The Definite Integral Represents Signed Area

Note that the value of $\int_a^b f(x)\,dx$ is not always the exact same number as the size of the area A between the function $f(x)$ and the x-axis. When the area in question is BELOW the x-axis it has a negative value.

Thus the definite integral $\int_a^b f(x)\,dx$ represents the **signed area** under the curve $f(x)$ on the interval of integration.

A. When $f(x) \geq 0$ over the interval of integration,

B. When $f(x) \leq 0$ over the interval of integration,

C. When $f(x)$ is sometimes positive and sometimes negative over the interval of integration,

One can compute a definite integral by knowing the area it represents.

EXAMPLE Find $\int_{-2}^{4} x\ dx$.

Properties of the Definite Integral

$$\begin{aligned}
\int_a^a f(x)\ dx &= 0 \\
\int_a^b f(x\ dx &= -\int_b^a f(x)\ dx \\
\int_a^b [f(x)+g(x)]\ dx &= \int_a^b f(x)\ dx + \int_a^b g(x)\ dx \\
\int_a^b [f(x)-g(x)]\ dx &= \int_a^b f(x)\ dx - \int_a^b g(x)\ dx \\
\int_a^b c \cdot f(x)\ dx &= c\int_a^b f(x)\ dx \\
\int_a^c f(x)\ dx + \int_c^b f(x)\ dx &= \int_a^b f(x)\ dx \\
\int_a^b f(x)\ dx &= -\int_b^a f(x)\ dx
\end{aligned}$$

Comparative Properties of the Definite Integral

If $f(x) \leq g(x)$, for all $x \in [a, b]$, then $\int_a^b f(x)\ dx \leq \int_a^b g(x)\ dx$

If $f(x) \geq 0$, for all $x \in [a, b]$, then $\int_a^b f(x)\ dx \geq 0$

If $m \leq f(x) \leq M$, for all $x \in [a, b]$, then $m(b-a) \leq \int_a^b f(x)\ dx \leq M(b-a)$