Math 120 Spring 2014

Names: __________________________ __________________________ __________________________

Lab #9
Math 120
Thursday
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Convergence Tests for Infinite Series

Today we are going to look at a few different tests for convergence of an infinite series of numbers. We will try to familiarize ourselves with methods which determine whether a particular infinite series converges.

SECTION A: Warm-Up

In *Derive* we can express a sum in two different ways.

I. One way that we can define a sum, whether infinite or finite, is to use the **C**alculus and **S**um options. Having chosen these options, the computer will ask you for a few pieces of information: (1) the expression to be summed (type in "k") – remember this represents a list of numbers, there is no "function variable" here; (2) the variable of summation – this is **NOT** a variable inside the sum, it is the variable which is "counting" through the integers for you (type in "k"); (3) the lower limit (type in "1"); (4) and the upper limit (**Tab** over and type in "3"). When you hit return you should see this sum written out for you. Write out this sum and calculate it **before** you ask the computer to do it.

$$\sum_{k=1}^{3} k =$$

Tell the computer to appro**X**imate it or **S**implify it. Do you get the same result? You should.

II. A second way that we can define a sum is by **A**uthoring the sum ourselves. **A**uthor the following:

SUM (k, k, 1, 3)

SUM (k^2, k, 1, 3)

Do you see the pattern? You can also **A**uthor a function involving a sum:

S(n) := SUM(k, k, 1, n)

Do you see what it does? What is S(3)? After you figure it out, **A**uthor S(3) and ask the computer to **S**implify it.

This last way of defining the sum as a function of n may seem tedious, but if you have to sum up the same thing over and over and over again, just changing the upper limit each time, then this is a great way to do it!

You can also ask Derive to solve the problem directly by putting in the sum an upper limit of ∞ by typing in "inf". Do this and see what the computer gives you, both symbolically and as an approximation. (Un)fortunately the computer will not always give you a meaningful answer if you put the upper limit as infinity, so you may have to use other methods to determine convergence of the series you are interested in.

Using `Wolfram|Alpha`, one can use similar commands at `www.wolframalpha.com` to investigate infinite series.

For example, consider typing the following expressions into `Wolfram|Alpha`

```
SUM ( k^2, k, 1, 3 )
```

and

```
SUM ( k^2, k, 1, n )
```

and

```
SUM ( k^2/(k+1), k, 1, n )
```

and

```
SUM ( k^2/(k+1), k, 1, inf )
```

and

What do your results tell you about the convergence of $\sum_{k=1}^{\infty} \frac{k^2}{k+1}$?
Which software do you prefer, `Derive` or `Wolfram|Alpha`? Why?

SECTION B: **Convergence Tests for Infinite Series**

I. n-th Term Test for Divergence

This test comes about from the definition of what convergence of a series means.

> If $\lim_{n\to\infty} a_n \neq 0$ (or does not exist) then the series $\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + a_4 + \ldots$ **diverges**.

Associated with the above fact is the idea that **IT IS ALWAYS TRUE** that if $\sum_{k=1}^{\infty} a_k$ CONVERGES, then $\lim_{n\to\infty} a_n = 0$.

We know that **IT IS NOT ALWAYS TRUE** that if $\lim_{n\to\infty} a_n = 0$ then $\sum_{k=1}^{\infty} a_k$ CONVERGES.

II. Integral Test for Convergence and Divergence

This test relates facts about improper integrals to facts about infinite series.

> Suppose $f(x)$ is a continuous and decreasing function and $f(x) > 0$ for all $x \geq 1$. Let $a(k) = f(k)$. THEN
> (a) If the $\int_1^{\infty} f(x)dx$ CONVERGES, then the infinite series $\sum_{k=1}^{\infty} a_k$ CONVERGES.
> (b) If the $\int_1^{\infty} f(x)dx$ DIVERGES, then the infinite series $\sum_{k=1}^{\infty} a_k$ DIVERGES.

Definition For $p > 0$ the infinite series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ is called a p-series.

> If one applies the integral test to the p-series then
> if $p \leq 1$, then p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ DIVERGES.
> If $p > 1$, then the p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ CONVERGES.

III. Comparison Test for Convergence and Divergence

(a) If $0 \leq b_k \leq a_k$ for each k and $\sum_{k=1}^{\infty} a_k$ converges, then $\sum_{k=1}^{\infty} b_k$ also CONVERGES.

(b) If $0 \leq a_k \leq c_k$ for each k and $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} c_k$ also DIVERGES.

IV. Alternating Series Test

Definition An infinite series is said to be an **alternating series** if it has the form $\sum_{k=1}^{\infty}(-1)^k a_k$ or $\sum_{k=1}^{\infty}(-1)^k a_k$ where $a_1, a_2, a_3, \ldots$ are all positive numbers.

If $a_1, a_2, a_3, \ldots, a_k, \ldots$ is a sequence of **decreasing positive numbers** such that $\lim_{n\to\infty} a_n = 0$ then the alternating series $\sum_{k=1}^{\infty}(-1)^{k+1} a_k$ CONVERGES.

V. Absolute Ratio Test

For any infinite series $\sum_{k=1}^{\infty} a_k$, if

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$$

then $\sum_{k=1}^{\infty} a_k$ CONVERGES.

If $L > 1$ or if $|a_{k+1}/a_k|$ does not exist, then $\sum_{k=1}^{\infty} a_k$ DIVERGES

If $L = 1$ the test is INCONCLUSIVE.

VI. Limit Comparison Test

Let $\sum_{k=1}^{\infty} a_k$, be an infinite series of **positive** terms.

(a) If $\sum_{k=1}^{\infty} c_k$ is a **convergent** series of positive terms, and $\lim_{n\to\infty} \frac{a_n}{c_n}$ EXISTS and is NOT INFINITE then $\sum_{k=1}^{\infty} a_k$ also CONVERGES.

(b) If $\sum_{k=1}^{\infty} c_k$ is a **divergent** series of positive terms, and $\lim_{n\to\infty} \frac{a_n}{d_n}$ EXISTS and is NOT ZERO or IS INFINITE then $\sum_{k=1}^{\infty} a_k$ also DIVERGES.

SECTION C: Examples Galore

Now that we have listed a number of tests for convergence the point of this lab is to have you consider the following infinite series and try and determine whether they converge or not.

Directions: For each of the following series, do the following:

- Write out the first few (about 3) **terms** of the series
- determine what kind of series it is (alternating, increasing, decreasing, positive, etc)
- Apply Test **I.** for Divergence on the series (n-th term test)
- See if `Derive` or `Wolfram|Alpha` can help you determine the convergence directly
- Determine convergence by using one of the other tests found in Section B

1. $\sum_{k=1}^{\infty} 4$

2. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$

3. $\sum_{k=1}^{\infty} \frac{2^k}{k!}$

4. $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$

5. $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k+5}}$

6. $\sum_{k=1}^{\infty} k^{1.4141}$

7. $\sum_{k=1}^{\infty} \frac{2}{k^2+1}$

8. $\sum_{k=1}^{\infty} \frac{2}{k^2}$

9. $\sum_{k=1}^{\infty} \frac{\ln(k+2)}{k^2}$

10. $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{k+1}$

11. $\sum_{k=1}^{\infty} ke^{-k}$

12. $\sum_{k=1}^{\infty} k \cos(k\pi)$

13. $\sum_{k=1}^{\infty} \sin\left(\frac{1}{k}\right)$

14. $\sum_{k=1}^{\infty} \sin\left(\frac{1}{k^2}\right)$

15. $\sum_{k=1}^{\infty} \frac{2^{k^2}}{k!}$

You should turn in one clean copy of this lab per group (with names and signatures of all group members on front page).