Introduction to Taylor Series
Class 29: Friday April 11

Infinite Series and Power Series
We have previously seen how to determine the convergence of infinite series. An infinite series is the sum of an infinite list (sequence) of numbers.

The reason this is useful is in determining when **POWER SERIES** converge. A power series is a sum of an infinite list (sequence) of polynomials, or a polynomial of infinite degree.

You can also think of a power series as an infinite series where the terms are powers of x.

1. What's a polynomial? Write down an example of a polynomial here: ______________________________

2. Therefore, write down an example of a power series here:______________________________

Taylor Series
The most useful form of a power series (and the only kind we will be looking at) is a **Taylor series**. A Taylor series, is just a **Taylor polynomial** of infinite degree.

Taylor Polynomials
They are used to APPROXIMATE a function near a particular point $x = a$. We use information about the function value and the derivative(s) of the function at the specific point $x = a$ to approximate the behavior of the function NEAR a

Approximating Functions Using Polynomials
For example, suppose we want to approximate $\sqrt{e} = e^{1/2}$ and we know $f(x) = e^x$ and its derivative $f'(x) = e^x$ at $x = 0$. That is, $f(0) =$ ______________, $f'(0)=$____________________.

3. From Calculus 1 we know we can write an approximation for $f(0.5) = e^{0.5}$ involving $f(0)$ and $f'(0)$

To do this problem you found the equation of the tangent line to the function $f(x) = e^x$ at $x = 0$. The equation of this tangent is called the **Taylor Polynomial of degree 1**
$e^x \approx P_1(x) = f(0) + f'(0)(x - 0)$

What would you do if you wanted to make the approximation *better*?

For example, we can compute Taylor polynomials of higher degree such as those given below for e^x based at $x = 0$:

$$\begin{aligned} e^x &\approx P_1(x) = 1 + x \\ e^x &\approx P_2(x) = 1 + x + \frac{x^2}{2!} \\ e^x &\approx P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \\ &\vdots \\ e^x &\approx P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^n}{n!} \end{aligned}$$

The **Taylor Series** for e^x about $x = 0$ can be written down below.
You can think of it as P_∞ or the Taylor Polynomial of infinite degree.

In general, the Taylor Series for a function $f(x)$ about the point $x = 0$ is

$$T(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k$$

NOTE

4. The difference between a Taylor **series** and a Taylor **polynomial** is

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The most general form of the Taylor Series for a function $f(x)$ about a point a is $\displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$

Example

5. Let's find $P_4(x)$ for the function $f(x) = \sin(x)$ about $x = 0$

GroupWork

6. Now find $P_4(x)$ for the function $f(x) = \sin(x^2)$ about $x = 0$

7. We can use this information to approximate the value of $\displaystyle\int_0^1 \sin(x^2)dx$