Improper Integrals
Class 22: Wednesday March 26

Definition

An integral is said to be an **Improper Integral** if one of the limits of integration is infinite or if the integrand becomes unbounded at some point in the interval of integration.

There are two types of improper integrals:

1. **Improper Integrals of the First Kind**

e.g. $\int_a^\infty f(x)\,dx$, $\int_{-\infty}^b f(x)\,dx$ or $\int_{-\infty}^\infty f(x)\,dx$

2. **Improper Integrals of the Second Kind**

$\int_a^b f(x)\,dx$, where either $\lim_{x\to a^+} f(x)=\infty$ or $\lim_{x\to b^-} f(x)=\infty$ or $\lim_{x\to c} f(x)=\infty$ when $a\le c\le b$

Evaluation

To evaluate an improper integral of the first kind one rewrites it as:

$$\begin{aligned}\int_a^\infty f(x)\,dx &= \lim_{b\to\infty}\int_a^b f(x)\,dx\\ \int_{-\infty}^b f(x)\,dx &= \lim_{a\to-\infty}\int_a^b f(x)\,dx\end{aligned}$$

If the limit exists, then the improper integral is said to *CONVERGE*.
If the limit is unbounded, the improper integral is said to *DIVERGE*.

To evaluate an improper integral of the second kind (assuming $f(x)$ is unbounded at some point $x=c$ where $a\le c\le b$):

$$\begin{aligned}\int_a^b f(x)\,dx &= \int_a^c f(x)\,dx+\int_c^b f(x)\,dx\\ &= \lim_{b\to c^-}\int_a^b f(x)\,dx \quad + \quad \lim_{a\to c^+}\int_a^b f(x)\,dx\end{aligned}$$

Similarly, if the limit exists, then the integral *CONVERGES*.
If the limit does not exist (or is unbounded) then the improper integral *DIVERGES*.

Understanding

In your own words, in the space below, write down a sentence which demonstrates your understanding of improper integrals:

In the space below draw pictures which graphically illustrate the two different types of improper integrals:

Trick Question

Look at the evaluation of the following integral. Is there anything wrong with this calculation? How would you do it differently? $I = \int_{-1}^{2} \frac{1}{x^2}\, dx = \left. \frac{-1}{x} \right|_{-1}^{2} = \frac{-1}{2} - \frac{-1}{-1} = \frac{-1}{2} - 1 = \frac{-1}{2}$

<u>Groupwork</u>

Evaluate the following integrals. If the integral is improper, say what KIND of improper integral it is and determine whether it CONVERGES or DIVERGES. Look for patterns, and try and discover the rules found on the last page.

1. $\displaystyle\int_2^\infty \frac{1}{x^{1.000001}}\,dx$

2. $\displaystyle\int_5^\infty \frac{1}{s^{0.99999999}}\,ds$

3. $\displaystyle\int_0^2 \frac{1}{t^4}\,dt$

4. $\displaystyle\int_1^5 \frac{1}{t-2}\,dt$

5. $\displaystyle\int_0^8 \frac{1}{\sqrt[3]{r}}\,dr$

6. $\displaystyle\int_0^3 x^{6/5}\,dx$

7. $\displaystyle\int_1^\infty e^{-2s}\,ds$

8. $\displaystyle\int_{-\infty}^1 e^{4r}\,dr$

Let's summarize our knowledge of improper integrals and limits.

$$\int_a^\infty \frac{dx}{x^p} = \begin{cases} \underline{\hspace{4cm}} & \text{when } p \leq 1 \\ \underline{\hspace{4cm}} & \text{when } p > 1 \end{cases}$$

$$\int_0^b \frac{dx}{x^p} = \begin{cases} \underline{\hspace{4cm}} & \text{when } p \geq 1 \\ \underline{\hspace{4cm}} & \text{when } p < 1 \end{cases}$$

$$\lim_{b\to\infty} b^p = \begin{cases} \underline{\hspace{4cm}} & \text{when } p < 0 \\ \underline{\hspace{4cm}} & \text{when } p > 0 \end{cases}$$

$$\lim_{x\to\infty} e^{kx} = \begin{cases} \underline{\hspace{4cm}} & \text{when } k < 0 \\ \underline{\hspace{4cm}} & \text{when } k > 0 \end{cases}$$