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Lab #2
Basic Calculus 2
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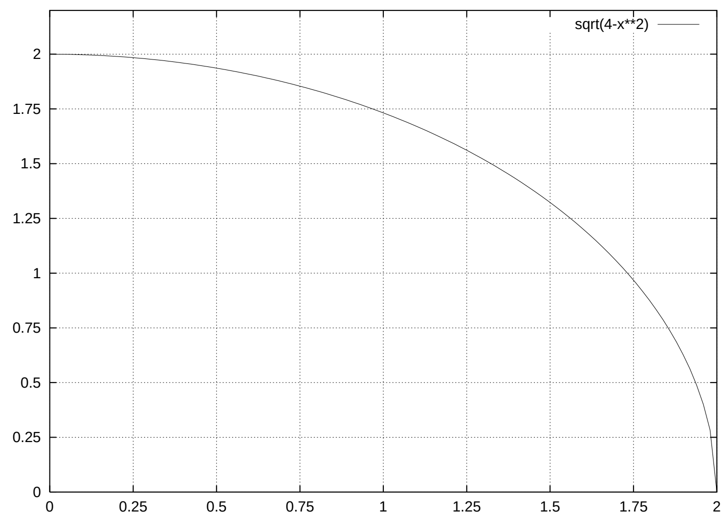
Estimating π as a Length

1. Setup. **TABLE WORK** $f(x) = \sqrt{4 - x^2}$ for $x \in [0, 2]$.



Recall that this curve is simply one quarter of the circle with radius $r = 2$ centered at $(0, 0)$. In the first lab, we saw that the area in the first quadrant under this curve was equal to π. Why is the *length* of the curve above also equal to π?

We can calculate lengths of curves by successive approximations using the distance formula. If we partition the interval $[0, 2]$ into N equal pieces, each with length Δx, and corresponding change in output value $\Delta y_k = f(t_{k-1}) - f(t_k)$, then

$$\text{LENGTH} \approx \sum_{k=1}^{N} \sqrt{(\Delta x)^2 + (\Delta y_k)^2}.$$

In order to verify this to the satisfaction of everyone in the group, on the picture above, break up the interval $[0, 2]$ into four pieces, and draw the four straight secant lines on the graph of the quarter circle you would use to approximate the length of the curve. Then write out and compute the sum $\sum_{k=1}^{4} \sqrt{(\Delta x)^2 + (\Delta y_k)^2}$.

2. $\boxed{\textbf{COMPUTER WORK}}$ The program **LENGTH.TRU** calculates the approximation on the previous page. Find this approximation using the program. You must adjust the program by defining the function f, giving the domain of the function, and telling it to use $n = 4$ steps. (The program is found by clicking on the **Mathematics** folder and then **Calculus** and then **Math120**.)

Now find the length of $f(x) = \sqrt{4 - x^2}$ on the interval $[0, 2]$ accurate to three decimal places. You will need to adjust the number of steps, n, until you converge to three decimal places of accuracy. Give your approximate length as well as how many steps you used to get this. (Note: Organizing your work into a table may help you see the convergence to three decimal places.)

3. $\boxed{\textbf{TABLE WORK}}$ **Converting to a Riemann Sum.** The sum $\sum_{k=1}^{N} \sqrt{(\Delta y_k)^2 + (\Delta x_k)^2}$ is NOT a Riemann sum. Do you know why?

We will know "convert" the sum $\sum_{k=1}^{N} \sqrt{(\Delta y_k)^2 + (\Delta x_k)^2}$ into a Riemann sum. In fact, we will use the microscope equation $\Delta y \approx f'(a)\Delta x$ to do so!

Fill in all the missing pieces below:

$$\text{LENGTH} \approx \sum_{k=1}^{N} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

• Since the microscope equation $\Delta y \approx f'(a)\Delta x$ tells us that for small Δx, the corresponding Δy is approximately equal to $f'(a)\Delta x$, we shall replace Δy_k with $f'(x_k)\Delta x$. Note that x_k is our choice of sampling point. Do so below:

$$\text{LENGTH} \approx \sum_{k=1}^{N} \sqrt{(\Delta x)^2 + (\qquad\qquad)^2}$$

• Factor $(\Delta x)^2$ out of the radical. Be careful to see you are removing it from both terms and be careful about what goes outside the radical now.

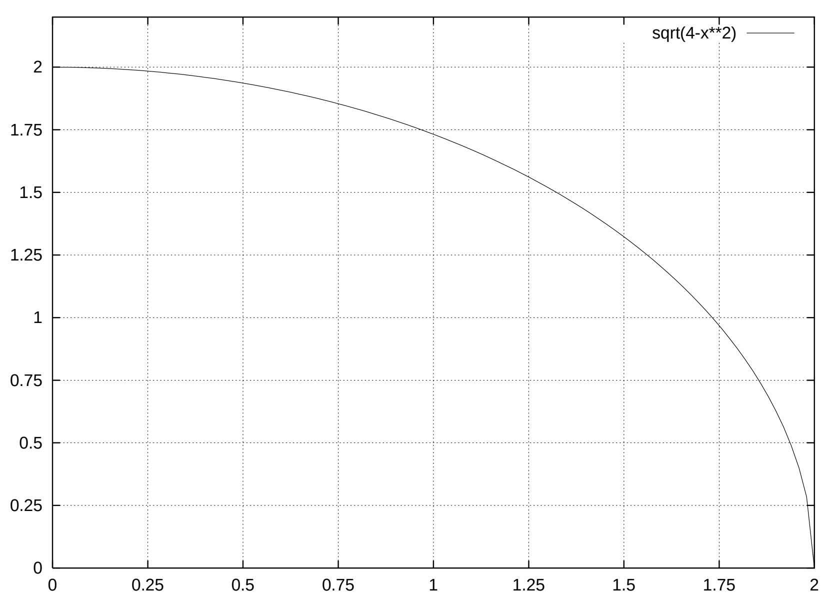
$$\text{LENGTH} \approx \sum_{k=1}^{N} \sqrt{(\quad) + (\qquad)^2} \quad (\qquad)$$

• You **should** now have a Riemann sum, $\sum_{k=1}^{N} g(x_k)\Delta x$, where the function we are summing is

$$g(x) = \sqrt{(\qquad\qquad\qquad)},$$

where f is the function, the arclength of which we want to find. Explain why we have converted it into the form of a Riemann sum. **REMARK:** Be sure you understand that this is a Riemann sum using g, **NOT** f itself.

Now we want to find a geometric interpretation for the sum $\sum_{k=1}^{N} g(x_k)\Delta x$. Below is another copy of the graph of $y = \sqrt{4 - x^2}$. As a group, come up with way of picturing why the sum above is an approximation of the arclength. Come up with a good explanation, as it will be a major part of your essay for this lab. Hint: derivatives go together with tangent lines.



We conclude that

$$\text{Length} = \lim_{N\to\infty} \sum_{k=1}^{N} \sqrt{1 + [f'(x_k)]^2}\Delta x = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

Before moving back to the computer, let's set up the Riemann sum which we obtained back in part 3 to find the length of $f(x) = \sqrt{4 - x^2}$ on the interval $[0, 2]$.

$f'(x) =$

$g(x) = \sqrt{[f'(x)]^2 + 1} =$

Simplify the function $g(x)$ above (using algebra) as much as possible before putting the function in the Riemann sum below:

$$\text{LENGTH} \approx \sum_{k=1}^{N} g(x_k)\Delta x_k =$$

4. $\boxed{\textbf{COMPUTER WORK}}$ Use **AGGSUM.TRU** (in the **Mathematics -¿ Calculus -¿ Math 120** subdirectory) with the appropriate function, $g(x)$ above, the appropriate interval, and the left endpoint sample points, $x_k = a + (k-1)\Delta x$ (or as the computer has it, LET x = a + k*Delta_x - (1)*Delta_x). (Hint: To save you some time, comment (!) out the line which prints each step.) Calculate this length to three decimal places accuracy. What do you get as the length and how many subintervals did this require? Why do you think it required such a fine partition (so many small steps)? (Hint: Look at the function you're taking a Riemann sum over (NOT f!).) Try running the program with sample points on the right of each interval, $x_k = a + k\Delta x$, and just four subintervals. What happens here and why did it happen?

WRITE-UP: Again, you should get together as a group later in the week, discuss the lab and the following questions: What is the difference between any old sum and a Riemann sum? Why couldn't our original method of approximating the length of a curve be considered a Riemann sum? Why was it so much harder for the Riemann sum to converge to three decimal place accuracy than for the sum using the distance formula? Why, when using right endpoints as your sampling points, did the the Riemann sum fail in the example done in lab? Will this always be the case, or does it depend on the function?

Write a brief, but clear essay (2-3 pages) comparing the two strategies for computing estimates of arclength. What are the advantages and disadvantages of each approach? This report is due in class **Friday, February 21, 2003**. You do NOT need to turn in this lab sheet, only the essay.