The Fundamental Theorem of Calculus

In our analysis of the problems of area, distance, work, and energy, we found that these quantities define accumulation functions. Today we will discover that the accumulation functions are solutions of initial value problems. The first step is to examine what the derivative of an accumulation function looks like in one of the familiar examples we've seen recently.

1. **Energy consumption.** We considered a power demand schedule $P = P(t)$, valid for the time interval staring at $t = a$ and ending at time $t = b$. Draw a graph of $P(t)$ below and follow the argument below by drawing some appropriate geometric figures on your graph.

For a small time interval of length Δt including time t, the energy consumed is given approximately by

$$\Delta E \approx P(t)\Delta t\,.$$

The entire amount of energy consumed from the starting time $t = a$ to the arbitary ending time $t = x$ is estimated by the Riemann sum

$$E(x) = \sum_{k=1}^{N} \Delta E_k \approx \sum_{k=1}^{N} P(t_k)\,\Delta t$$

(using $\Delta t = \frac{1}{N}(x - a)$). The approximation converges as N increases without bound. The exact value of the energy consumed is given by the integral

$$E(x) = \int_0^x P(t)\,dt.$$

Now we want to examine the derivative of $E(x)$. What is the appropriate geometric interpretation *on the graph* of $P(t)$ of the expression $E(x + \Delta x) - E(x)$?

How can we use the function $P(t)$ to estimate the area of the region we just determined?

We have two expressions (one approximate) for this area. If we divide both of them by Δx and let $\Delta x \to 0$, the values of these expressions approach each other, and in the limit we have

$$E'(x) = \frac{dE}{dx} = \lim_{\Delta x \to 0} \frac{\Delta E}{\Delta x} = P(x),$$
$$E(a) = 0.$$

(Where does the *initial condition* $E(a) = 0$ come from?)

The general statment.

The pattern of these results is striking. A quantity determined by accumulation can be expressed as an integral. Provided the accumulation function is differentiable, its derivative is easily expressed. This result is known as the **Fundamental Theorem of Calculus**.

If f is a continuous function on the interval $[a, b]$ and we let

$$A(x) = \int_a^x f(t)\,dt, \quad a \le x \le b$$

then $A(x)$ is differentiable and

$$A'(x) = f(x) \quad \textit{and} \quad A(a) = 0.$$

What is "fundamental" is that a quantity which can be expressed as an accumulation can also be expressed as the solution to an initial value problem.

Using the ideas in the examples we examined today, can you prove the Fundamental Theorem of Calculus?

Using the Fundamental Theorem

The Fundamental Theorem connects the two key ideas of calculus–rate of change and accumulation–and can be used to import knowledge about one idea into the solution of a problem which occurs with the other one. This theorem has different versions corresponding to different ways it can be used.

Derivative of an accumulation. An accumulation function is defined as an integral with a *variable* endpoint of integration:

$$A(x) = \int_a^x f(t)\,dt$$

The fundamental theorem tells us about the derivative of the accumulation function. This exists if $f(t)$ is continuous, in which case:

$$A'(x) = f(x)$$

Accumulation functions are antiderivatives. Because of the Fundamental Theorem, we have the following compact equation for *continuous* $f(t)$:

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

Explain how this follows from the Fundamental Theorem.

Accumulation functions solve initial value problems. The initial value problem

$$y'(x) = f(x)$$
$$y(a) = C$$

where $f(x)$ is continuous and C is a constant, has the unique solution

$$y(x) = C + \int_a^x f(t)\,dt$$

Explain how this follows from the Fundamental Theorem.

Problems for Practice

Use the Fundamental Theorem of Calculus to quickly solve the following problems:

A. $\frac{d}{dx}\left(\int_0^x e^{-t^2}\,dt\right) =$

B. Solve the initial value problem:

$$y'(x) = x$$
$$y(0) = 2$$