Riemann Sums and The Definite Integral

We have been using Riemann sums to estimate the solution to problems like the following:

Distance Travelled

A driver applies the brakes, bringing the car to a halt in 6 seconds. Suppose the speed $v(t)$ of the car is initially 100 ft/sec and declines as the brakes are applied. If the speed is recorded at 1 sec intervals after braking begins, write sums for an underestimate and an overestimate of the distance travelled by the car during these 6 seconds.

$$< \textit{Distance} <$$

As the number of times the speed is recorded increases so that the interval between recording times decreases, what happens to the difference between the corresponding over- and underestimating sums?

Why?

Riemann Sums

Suppose a function f is defined on an interval $a \le t \le b$. Suppose the interval is divided up into n subintervals of lengths $\Delta t_1, \ldots, \Delta t_n$. Suppose t_k is a point chosen from the subinterval Δt_k for each $k = 1, 2, \ldots, n$. A **Riemann sum** for f on the interval $[a, b]$ is a sum of the form

$$\sum_{k=1}^{n} f(t_k)\Delta t_k = f(t_1)\Delta t_1 + f(t_2)\Delta t_2 + \cdots + f(t_n)\Delta t_n.$$

If the *left* endpoint of each subinterval is chosen, the Riemann sum is a **left-hand sum**.

If the *right* endpoint of each subinterval is chosen, the Riemann sum is a **right-hand sum**.

In making an estimate in the form of a Riemann sum, why might you want subintervals of different sizes?

Why might you want to be able to choose the sample point t_k anywhere within the subinterval Δt_k?

To improve the approximation given by a Riemann sum, you will want to let the number n of subintervals increase, taking the limit as $n \to \infty$. As you do so, you will also want to make sure that

$$\lim_{n\to\infty} \max_{k=1,\ldots,n} \{\Delta t_k\} = 0.$$

Why?

Definition: (Definite Integral)

Suppose $[a, b]$ is a bounded interval and f is bounded on $[a, b]$. Given n, suppose $[a, b]$ is subdivided into n subintervals of lengths $\Delta t_1, \ldots, \Delta t_n$. Let t_k be a sample point in the k^{th} subinterval. Then the **definite (Riemann) integral** of f on $[a, b]$ is defined as

$$\int_a^b f(t)dt := \lim_{n\to\infty} \sum_{k=1}^{n} f(t_k)\,\Delta t_k,$$

provided this limit exists, does not depend on how sample points are chosen within subintervals, and is taken so that the subinterval lengths shrink to zero as $n \to \infty$. If this integral exists, f is said to be *(Riemann) integrable* on $[a, b]$. The number "a" is said to be the *lower limit* of the integral and "b" is said to be its *upper limit*.

Use the definite integral notation to write the distance travelled by the car in terms of the car's speed.

Representation of a Definite Integral as "Signed Areas Under the Curve"

So far we have examined sums associated with positive functions, that is, functions that always assume positive values. We have seen if $f(x) \geq 0$ over an interval of interest, then the area between the graph of f and that interval on the axis can be estimated by summing of terms of the form $f(x) * \Delta x$. This sum is a Riemann sum for f, and since in the appropriate limit, the Riemann sum approaches both the area under the curve and the definite integral, we may interpret the definite integral as an area under a curve.

However, our definition of the definite integral did not require that the function $f(x)$ be positive. At times we need to consider functions which are entirely *negative* or which assume both positive and negative values. In such cases, we may still represent the definite integral as an area, but we must add the qualification that it is a *signed* area:

If a function is always negative, then its definite integral is negative. In fact, the absolute value of the definite integral is precisely the area trapped between the graph of the function $f(x)$ and the x-axis. The the negative sign indicates that the region is *below* the x-axis.

When $f(x)$ is positive for some values of x and negative for others, and $a < b$, then

$$\int_a^b f(x)dx$$

is the sum of the areas above the x-axis, counted positively, and the areas below the x-axis, counted negatively.

Example: Use this geometric interpretation to show that $\displaystyle\int_{-a}^{a} x dx = 0$.

Example: Use the geometric interpretation of the definite integral, along with a graph of $x^3 - 1$, to determine whether $\displaystyle\int_{-1}^{2} (x^3 - 1)dx$ is positive or negative.