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Lab #8
Math 120 Lab
Wednesday/Thursday
April 4/5, 2001
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# Evaluating and estimating improper integrals

This lab develops two methods for determining the value of an "improper integral of the first kind"

$$\int_a^\infty f(x)\,dx := \lim_{b\to\infty}\int_a^b f(x)\,dx$$

One method can determine an exact value for the limit; the other method gives a numerical estimate.

## An exact value for an improper integral

When an antiderivative can be determined for a function $f(x)$, it is sometimes possible to determine the exact value for the improper integral

$$\int_a^\infty f(x)\,dx$$

This section develops one such example. The *method* used here can be applied to other examples.

1. Give the calculations to show that $\int_0^{100}\left(\frac{1}{2}\right)^x\,dx = \frac{1}{\ln 2}\left(1-\frac{1}{2^{100}}\right)$.

2. By how much does your answer above differ from $\frac{1}{\ln 2}$? Express this number two ways: exactly and as a decimal approximation.

3. Give an expression for $\int_0^{1000} \left(\frac{1}{2}\right)^x \, dx$. (Refer to earlier work; you do not have to show all the steps here.)

4. Give an exact expression for $\int_0^b \left(\frac{1}{2}\right)^x \, dx$ using the variable $b$. (Show all the steps this time.)

5. How did the earlier work help you answer the last question?

6. Determine the exact value of the improper integral

$$\int_0^{\infty} \left(\frac{1}{2}\right)^x \, dx.$$

7. What is the area of the region in the first quadrant of the $xy$-coordinate plane bounded above by the graph $y = \left(\frac{1}{2}\right)^x$ ?

8. (This result here will be used below.) Determine the exact value of the improper integral $\int_{20}^{\infty} \left(\frac{1}{2}\right)^x dx$. Also, give a decimal approximation. Show all the steps in your work.

**An estimate for an improper integral**

Unlike the example in the previous section, the antiderivative $\int \left(\frac{1}{2}\right)^{x^2} dx$ cannot be expressed as an ordinary formula.[1] In this section, you can give a good approximation for the value of the improper integral $\int_0^\infty \left(\frac{1}{2}\right)^{x^2} dx$. The strategy is to use the integral property

$$\int_0^\infty \left(\frac{1}{2}\right)^{x^2} dx = \int_0^c \left(\frac{1}{2}\right)^{x^2} dx + \int_c^\infty \left(\frac{1}{2}\right)^{x^2} dx$$

to guide the analysis. A numerical estimate can be obtained for the first term of the sum; then, we can show that the second term of the sum is very small.

9. Use the TRUE BASIC program SIMPSON to give successive approximations to $\int_0^{20} \left(\frac{1}{2}\right)^{x^2} dx$. Record at least five approximations and continue until you are confident that you have at least three decimal places of accuracy in the estimate.

[1] Some computer algebra systems, like DERIVE, will express this antiderivative in terms of the "error function" $ERF(x)$ defined by the integral $ERF(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt$.

10. Explain why $\int_a^b \left(\frac{1}{2}\right)^{x^2} dx < \int_a^b \left(\frac{1}{2}\right)^{x} dx$ for any interval $[a,b]$ with $a > 1$. *(Hint: sketch a diagram of the graphs.)*

11. Use the previous result to show that

$$\int_{20}^{\infty} \left(\frac{1}{2}\right)^{x^2} dx < 0.00001$$

(This estimate is sometimes called "bounding the tail" of the improper integral.)

12. Give a rough sketch of the graph $y = \left(\frac{1}{2}\right)^{x^2}$ for $x > 0$. Break the region under the curve at $x = 20$. What is the area of the left-hand portion of the region? What can you say for sure about the area of the right-hand portion?

13. Give a three-place estimate of the improper integral $\displaystyle\int_0^{\infty} \left(\frac{1}{2}\right)^{x^2} dx$. Use the results of previous computations.

14. How could you adapt the method used here to improve the accuracy of the estimate of the improper integral? Write a paragraph to describe what you would do.

15. *(Extra credit.)* Following the strategy you just outlined, give a six-place estimate to the same improper integral.

**Preparing Your Lab Report**

As before, your report should consist of a cover page with the title of the report and the names and signatures of your lab group members. **Also indicate your Lab Section and Lab Time and Lab Professor.** Each person (in a group of three) should draft one of the three parts, and the group should meet to read and comment on these drafts before submitting the final report.

Your lab report should contain full answers to the following questions:

*Part 1*
Questions 4–8 above.

*Part 2*
Questions 9–12 above.

*Part 3*
Questions 13–15 above.

**Due Date**
Your lab report is due **Wednesday/Thursday April 11/12**, in lab. You should plan to meet and share progress on the report with your lab group partners throughout the week. You may consult your professors for guidance on your work.

**Warm up problems for Lab 7**

Each of these problems will be used in the work in later sections. Please check your solution and your understanding with your group members.

1. What is the relationship between $\ln(a)$ and $\ln(\frac{1}{a})$? Illustrate the relationship numerically for two values of $a$.

2. Sketch the graph of $y = (\frac{1}{2})^x$, for $x > 0$. Shade in the region under the graph. How far to the right does the region extend (in principle)?

3. Give a formula for the antiderivative $\int (\frac{1}{2})^x \, dx$. (Use the natural logarithm function in your expression, rather than a decimal approximation.)

4. For $x > 1$, which is larger: $(\frac{1}{2})^x$ or $(\frac{1}{2})^{x^2}$ ? Explain why you are sure; you can give an algebraic or a geometric argument.