Class 12: Friday September 26

Functions of Several Variables

Introduction

We generally understand the relationship between the derivatives of a function $y = f(x)$ and the shape of the graph. This allows us to find the *extrema* of a function, i.e. where it has a global or absolute maximum (or minimum) value, without ever graphing it. This idea is so important there is a whole branch of mathematics devoted to it known as **optimization**.

Our goal in this next section of the class is to try and learn how to find extrema of functions of more than one variable, i.e. $z = f(x, y)$

Functions of Two Variables

DEFINITION:

Suppose the value of an output variable z is *uniquely* determined once the values of two input variable x and y are given. Then we say that z **is a function of** x **and** y.

As with functions of one variable, to specify the function we need to specify the *domain*, *co-domain*, *name of the function*, and the *rule* assigning a *unique* output value to *each* input pair.

Example

$$f : R^2 \to R, \quad z = f(x, y) = 2x^2 + 3xy - 4$$

means that the function *named* f has the set R^2 of *pairs* of real numbers as its domain and the set R of real numbers as its *co-domain*. The ordered pair of input variables is named (x, y). The output variable is named z. The rule which assigns a value to z given the values of x and y is

$$z = f(x, y) = 2x^2 + 3xy - 4.$$

For this example...

1. If $(x, y) = (1, 2)$, what is the value of z?

2. Can you find a value for x and a value for y which make $z = -4$? Can you find more than one such pair of values? Why doesn't this contradict the claim that f is a *function* (of two variables)?

Calculating Partial Derivatives

Fortunately, we rarely actually have to evaluate a limit to compute a partial derivative. Your knowledge of derivatives for functions of one variable is enough! Therefore,

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h} = \lim_{h\to 0} \frac{g(a+h) - g(a)}{h} = g'(a).$$

TO COMPUTE $f_x(a,b)$, JUST DIFFERENTIATE $f(x,y)$ WITH RESPECT TO x, regarding it as a function of x only (i.e. TREATING y AS IF IT WERE A CONSTANT), then evaluate the result at $x = a$, $y = b$.

Similarly, we defined the vertical slice of f along y, holding $x = a$, as $\phi(y) = f(a,y)$. Therefore,

$$f_y(a,b) = \lim_{k\to 0} \frac{f(a,b+k) - f(a,b)}{k} = \lim_{k\to 0} \frac{\phi(b+k) - \phi(b)}{k} = \phi'(b).$$

TO COMPUTE $f_y(a,b)$, JUST DIFFERENTIATE $f(x,y)$ WITH RESPECT TO y, regarding it as a function of y only (i.e. TREATING x AS IF IT WERE A CONSTANT), then evaluate the result at $x = a$, $y = b$.

Example: For $f : R^2 \to R, \quad z = f(x,y) = 2x^2 + 3xy - 4$ ...

3. Find the following partial derivatives:

$f_x(2,3)$

$f_y(2,3)$

$f_x(a,b)$

$f_y(a,b)$

$f_x(x,y)$

$f_y(x,y)$