Class 7: Monday September 15

Population Growth Models and Steady States

Population Models: Constant Growth, Exponential Growth, Logistic Growth

Constant Growth

The first model for a population growth could be that it is growing at a constant rate, i.e. $P' = k$. What is unrealistic about this model?

Sketch a graph of $P_c(t)$ versus t.

Exponential Growth

$$P' = kP, \qquad P(0) = P_0 \tag{1}$$

The Bacteria (1) model shows how bacteria grow in the absence of any constraints. That is, every bacteria is well-fed and nothing intervenes to kill the bacteria. Under these conditions, each bacteria divides at regular intervals, so the growth rate is proportional to the number of bacteria ($P' = kP$). Another way to say this is that the growth per bacterium is constant, or $P'/P = k$. The ratio P'/P is known as the relative rate of growth. The relative growth rate is scaled to give a better sense of how fast something is growing for its size. What is unrealistic about this model?

Sketch a graph of $P_e(t)$ versus t.

Logistic Growth

$$P' = kP\left(1 - \frac{P}{M}\right), \qquad P(0) = P_0 \tag{2}$$

This model repeats the bacterial growth model with an adjustment that constrains the growth rate due to competition for resources, that is, overcrowding.
In the new bacteria model we suppose that there is enough room for M bacteria. Thus the fraction of the space already occupied by the bacteria (presumably equal to the fraction of the resources that the bacteria use, since they don't build hiqhways,) is P/M and the fraction of the unused space is $1 - P/M$.
In this model we assume that the relative growth rate for the bacteria population is proportional to the amount of unused space. If $P'/P = k(1 - \frac{P}{M})$, then $P' = kP(1 - P/M)$. What is unrealistic about this model?

Sketch a graph of $P_l(t)$ versus t above.

1. How do the graphs of the solution of each of these three models differ over time? (Mathematically, what happens as $t \to \infty$)

2. Which model is the most realistic? Is there ever a situation where the population decreases with time?

Obtaining Information from an IVP without solving it

Consider the general form of an IVP: $y' = f(t,y), \quad y(a) = b$

Obtaining steady state behavior from an IVP.

What information do we have about the solution $y(t)$ when $f' > 0$? What about when $f' < 0$? $f' = 0$?

A **steady state** of a model is said to exist when the rate of change of the solution to the differential equation is zero. In other words, regardless of the change in the independent variable t, the dependent variable y is constant or $f' = 0$.

Obtaining concavity information from an IVP.

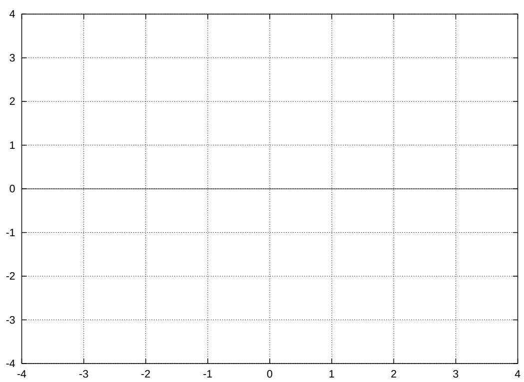
The differential equation $y' = f(t,y)$ tells us exactly how the solution $y(t)$ is changing with respect to t and y. Thus we can know for what values the solution curves are increasing or decreasing. Depending on the form of $f(t,y)$ we can obtain expressions for y'' from the IVP and thus be able to say something about the concavity of the solution curves without being able to solve the IVP exactly.

Examples

1. Consider the differential equation $y' = t^2$

a. For what values is $y' = 0$? What is the steady state of the model?

b. Can you sketch solutions to the model using the initial conditions $y(0) = -1$ or $y(0) = 1$ below? (HINT: What is y''?)

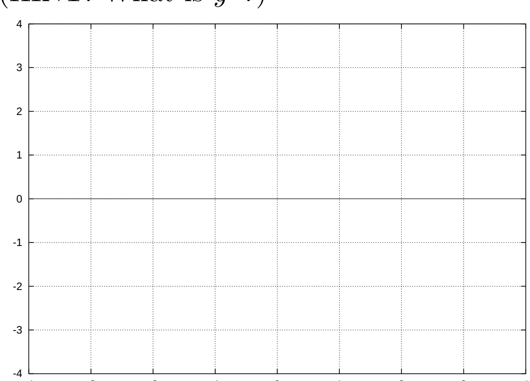


c. How do your sketches compare to the exact solutions to the IVPs?

2. Consider the differential equation $y' = 2y$

a. For what values is $y' = 0$? What is the steady state of the model?

b. Can you sketch solutions to the model using the initial conditions $y(0) = -1$ or $y(0) = 1$ below? (HINT: What is y''?)

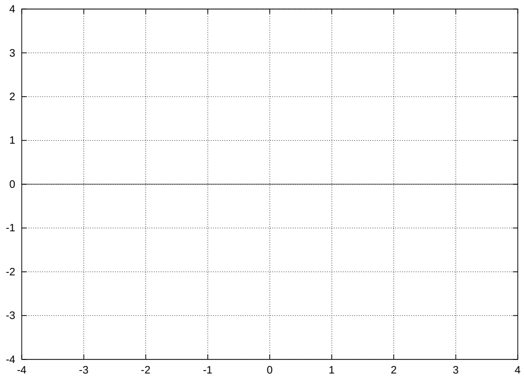


c. How do your sketches compare to the exact solutions to the IVPs?

3. Consider the differential equation $y' = 2 - y$

a. For what values is $y' = 0$? What is the steady state of the model?

b. Can you sketch solutions to the model using the initial conditions $y(0) = -1$ or $y(0) = 1$ below? (HINT: What is y''?) **Also sketch solutions using** $y(0) = 3$

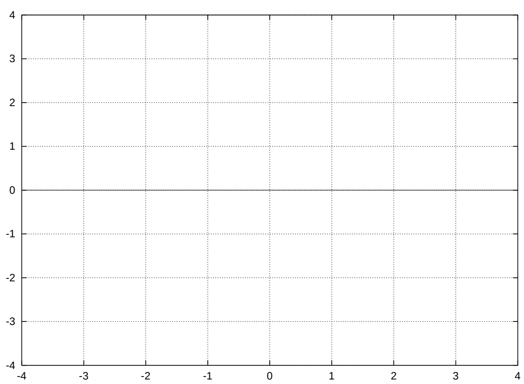


c. How do your sketches compare to the exact solutions to the IVPs?

4. Consider the differential equation $y' = ty$

a. For what values is $y' = 0$? What is the steady state of the model?

b. Can you sketch solutions to the model using the initial conditions $y(0) = -1$ or $y(0) = 1$ below? (HINT: What is y''?)

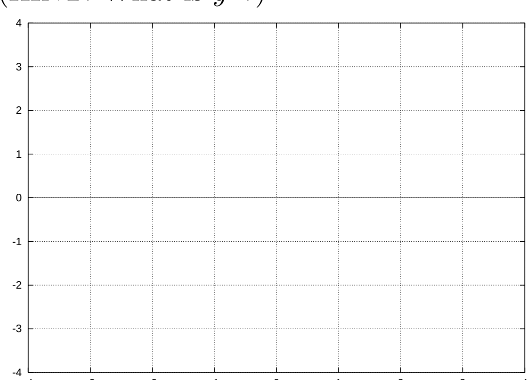


c. How do your sketches compare to the exact solutions to the IVPs?

5. Consider the differential equation $y' = -ty$

a. For what values is $y' = 0$? What is the steady state of the model?

b. Can you sketch solutions to the model using the initial conditions $y(0) = -1$ or $y(0) = 1$ below? (HINT: What is y''?)



c. How do your sketches compare to the exact solutions to the IVPs?