Review of Limits

We have seen that calculations of successive approximations to a value can "stabilize" around a limiting value. We have also seen that in some models, the values of functions for increasingly large inputs can stabilize.

Examples

1. For successive Euler's method calculations of a value like $Y(3)$ using various Δt values, the results become "stable."
2. In both our epidemic and Newton's law of cooling models, as time increases, there is less and less variation in the values of the variables. For example, as time goes on in the SIR model, the number of infected persons gets closer to 0.
3. In analyzing one step of Euler's method applied to a rate equation $y' = G(x, y)$, we use

$$\Delta y \approx y' \cdot \Delta x, \quad \text{for } \Delta x \approx 0$$

Another way to say this is

$$y' \approx \frac{\Delta y}{\Delta x}, \quad \text{for } \Delta x \approx 0$$

Expression as limits.

Each of these examples is often stated as a "limit."

1. $\lim_{\Delta t \to 0} \big(\text{Calculated value of } Y(3)\big) = \text{The temperature after 3 minutes.}$
2. $\lim_{t \to \infty} I(t) = 0$
3. $y' = \lim_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$

Sequences and Limits: *Question:* What is a sequence?

Write the first few terms of the following sequences:

1. $a_k = k$; for $k \geq 0$

2. $a_k = \frac{1}{2^k}$; for $k = 0, 1, 2, \ldots$

Do either of these sequences get closer and closer to some value?

Sidenote: Recall that $|t-a| < b$ means $-b < t-a < b$. This is equivalent to: $a-b < t < a+b$ or to the statement $t \in (a-b, a+b)$.
Use absolute values to rewrite the following statements:

1. $3-2 < t < 3+2$
2. $6 < t < 10$
3. $t \in (0,5)$

Convergence of a Sequence: A sequence converges to a limit if its terms get closer and closer to the limit value. Using absolute values can make this idea more precise.

A sequence $\{a_k\}, k = 0, 1, 2, \ldots,$ is said to *converge* to the *limit* a, written

$$\lim_{k\to\infty} a_k = a \qquad \text{or} \qquad a_k \to a \quad \text{as} \quad k \to \infty,$$

if the *absolute error* $|a_k - a|$ between a_k and a can be made as small as you wish by taking k to be large enough.

A sequence which does not converge is said to **diverge**.

Example

1. Suppose $a_k = (0.1)^k$. As $k \to \infty$, we expect $a_k \to 0$.
2. Test this by first picking a small number, say 0.0001.
3. Then **find a value of** N so that

$$|a_k - a| = |(0.1)^k - 0| \le 0.0001 \quad \text{for } k \ge N =$$

Do you think you could find such an N given any particular small size for the error? Try to do so. Suppose you let E denote the largest error you will tolerate. Can you find a rule for N, which depends on E, so that

$$|a_k - a| = |(0.1)^k - 0| \le E \quad \text{for } k \ge N =$$

Summary: Essentially, if a sequence approaches a limit, then you can always find at least one term of the sequence **as close to the limit value as you want**; this is true whether you want the term as close as 1 (which is not really very close) or as close as 10^{-25} (which is really, really close) to the limit.

Convergence of functions to a limit

The language of limits that applies to sequences also applies to functions.

A function $y = f(t)$ is said to *converge* to the *finite limit* L as t approaches a **from below**, written

$$\lim_{t \to a^-} f(t) = L,$$

if $\lim_{k \to \infty} f(t_k) = L$ for *any* sequence $t_k \to a$ such that $t_k < a$ for all k.

A function $y = f(t)$ is said to *converge* to the *finite limit* L as t approaches a **from above**, written

$$\lim_{t \to a^+} f(t) = L,$$

if $\lim_{k \to \infty} f(t_k) = L$ for *any* sequence $t_k \to a$ such that $t_k > a$ for all k.

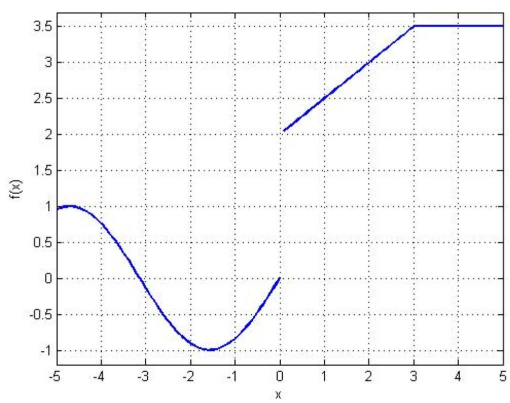
A function $y = f(t)$ is said to **converge to the finite limit** L as t approaches a, written

$$\lim_{t \to a} f(t) = L,$$

if $\lim_{t \to a^-} f(t)$ and $\lim_{t \to a^+} f(t)$ both exist, and if

$$\lim_{t \to a^-} f(t) = L = \lim_{t \to a^+} f(t).$$

Graphical Convergence of a Function to a Limit



Using the figure above, answer the following questions.

(a) What is $\lim_{x \to 0^-} f(x)$?

(b) What is $\lim_{x \to 0^+} f(x)$?

(c) What is $\lim_{x \to 3^-} f(x)$?

(d) What is $\lim_{x \to 3^+} f(x)$?

(e) What are $\lim_{x \to 0} f(x)$ and $\lim_{x \to 3} f(x)$?

Rules of Limits

The following rules of limits apply (to both limits of functions and limits of sequences).
In the rules below, L and M are finite.

1. **"The limit of a sum is the sum of the limits."**
If $\lim_{t\to a} f(t) = L$ and $\lim_{t\to a} g(t) = M$, then $\lim_{t\to a} f(t) + g(t) = L + M$.

2. **"The limit of a product is the product of the limits."**
If $\lim_{t\to a} f(t) = L$ and $\lim_{t\to a} g(t) = M$, then $\lim_{t\to a} f(t) \cdot g(t) = L \cdot M$.

3. **"The limit of a ratio is the ratio of the limits, provided the denominator does not converge to zero."**
If $\lim_{t\to a} f(t) = L$ and $\lim_{t\to a} g(t) = M \neq 0$, then $\lim_{t\to a} f(t)/g(t) = L/M$.

Practice with Limits: Evaluate the following limits.

a. $\lim_{x\to 0}(x^2 - 3x + 1)$

b. $\lim_{k\to 2} \frac{k+3}{k+2}$

c. $\lim_{t\to -2} \frac{t+2}{t^2-4}$

d. $\lim_{\theta\to 0} \frac{\sin\theta}{\tan\theta}$

e. $\lim_{k\to\infty} 2^{-k}$

f. $\lim_{\Delta x\to 0} \frac{\sqrt{4+\Delta x} - 2}{\Delta x}$