Multivariable Optimization

Optimization for Functions of Two Variables

Definition:

- $z = f(x, y)$ has a **local maximum** at (x_0, y_0) if $f(x_0, y_0) \geq f(x, y)$, for all (x, y) in some neighborhood of (x_0, y_0).
- Similarly, $z = f(x, y)$ has a **local minimum** at (x_0, y_0) if $f(x_0, y_0) \leq f(x, y)$, for all (x, y) in some neighborhood of (x_0, y_0).

Critical Points

1. Suppose we take a vertical slice in the x-direction through $f(x, y)$ at a local maximum (x_0, y_0), and suppose the partial derivatives of f exist there. What will this slice look like near this point? What is the value of $f_x(x_0, y_0)$?

2. Suppose we take a vertical slice in the y-direction through $f(x, y)$ at a local maximum (x_0, y_0), and suppose the partial derivatives of f exist there. What will this slice look like near this point? What is the value of $f_y(x_0, y_0)$?

Definition: Suppose $f(x, y)$ and its partial derivatives exist in a neighborhood of (x_0, y_0). Then (x_0, y_0) is a **critical point** for f if

$$f_x(x_0, y_0) = 0 \quad \text{and} \quad f_y(x_0, y_0) = 0.$$

Critical points are candidates for local maxima and local minima. Critical points which are not local maxima or local minima are called **saddle points** (a saddle point is similar to an inflection point). Contour plots can be helpful in classifying critical points.

Examples

4. Find critical points for the following functions:

a) $g(x, y) = x^2 + y^2$

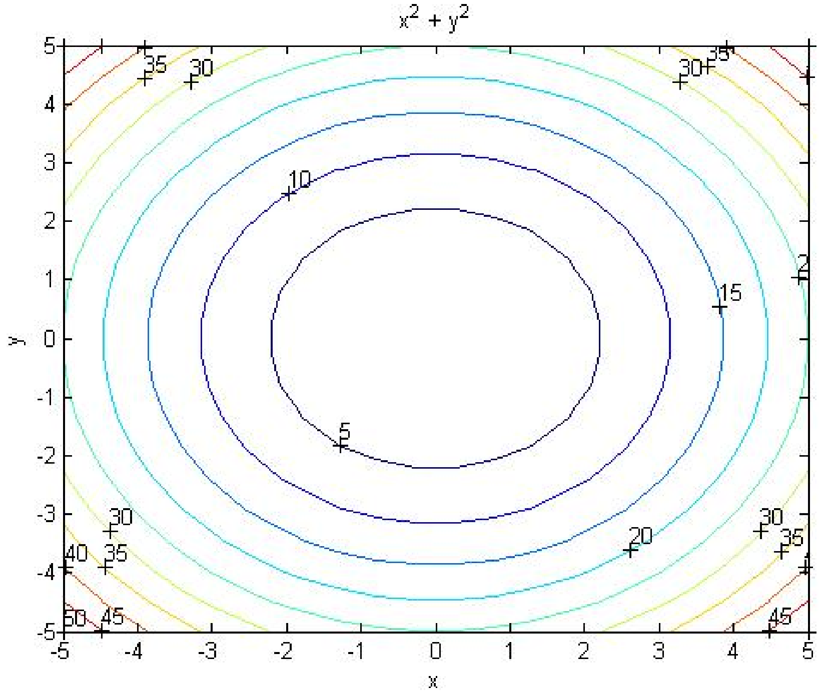
b) $j(x, y) = x^2 - y^2$

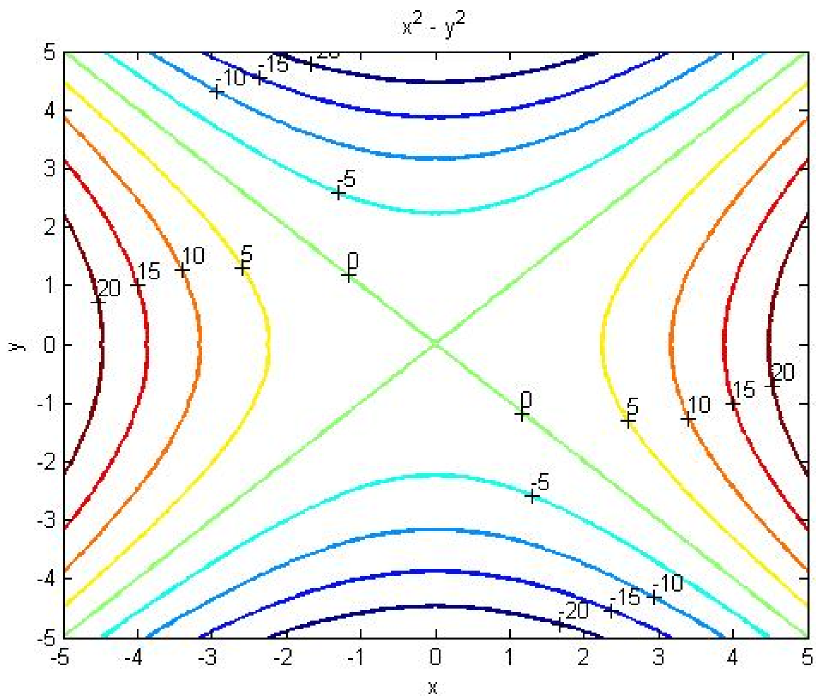
c) $f(x, y) = (x + y)^2$

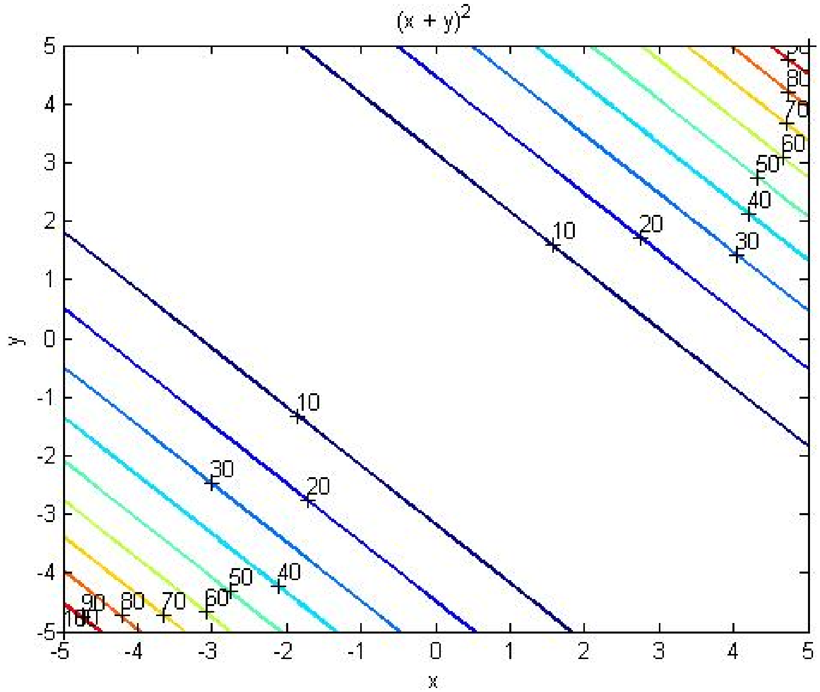
d) $k(x, y) = \sin(x) - \sin(y)$

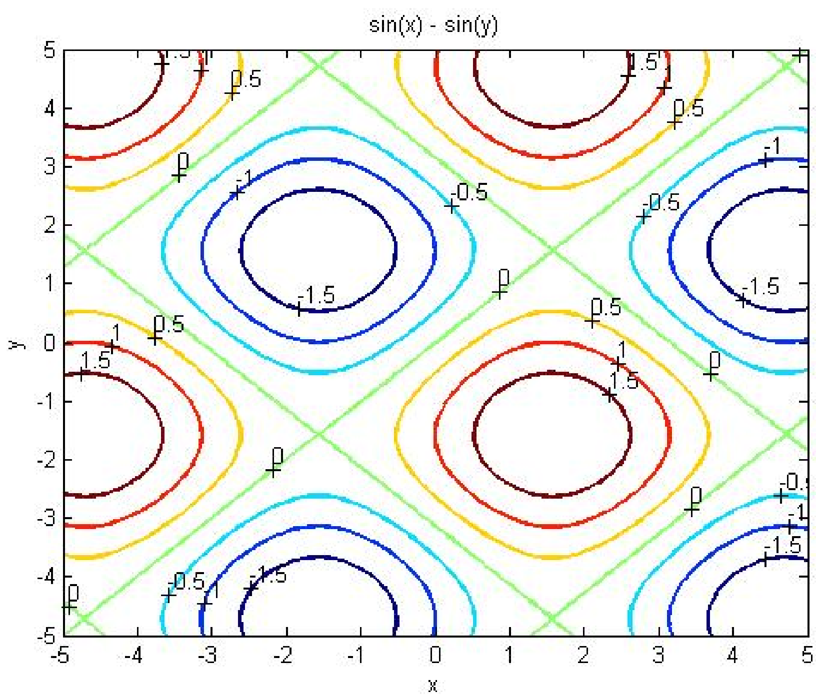
Match these functions to the contour plots on the following page, and classify the critical points as local maxima, local minima, or saddle points.

Contour Plots






(x + y)²
y
x
10
20
30
40
50
60
70
80
90


sin(x) - sin(y)
y
x
0
0.5
1
1.5
-0.5
-1
-1.5