Functions of Two Variables

Introduction
We generally understand the relationship between the derivatives of a function $y = f(x)$ and the shape of the graph of $f(x)$. This allows us to find the *extrema* of the function, i.e. where it has a global or absolute maximum (or minimum) value, without ever graphing it. This idea is so important there is a whole branch of mathematics devoted to it known as **optimization**.

Our goal in this last section of the class is to try and learn how to find extrema of functions of more than one variable, i.e. $z = f(x, y)$

Functions of Two Variables

DEFINITION:

Suppose the value of an output variable z is *uniquely* determined once the values of two input variable x and y are given. Then we say that z **is a function of** x **and** y.

As with functions of one variable, to specify the function we need to specify the *domain*, *range*, *name of the function*, and the *rule* assigning a *unique* output value to *each* input pair.

Example

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad z = f(x, y) = 2x^2 + 3xy - 4$$

means that the function *named* f has the set $\mathbb{R}^2$ of *pairs* of real numbers as its domain and the set $\mathbb{R}$ of real numbers as its *range*. The ordered pair of input variables is named (x, y). The output variable is named z. The rule which assigns a value to z given the values of x and y is

$$z = f(x, y) = 2x^2 + 3xy - 4.$$

For this example...

1. If $(x, y) = (1, 2)$, what is the value of z?

2. Can you find a value for x and a value for y which make $z = -4$? Can you find more than one such pair of values? Why doesn't this contradict the claim that f is a *function* (of two variables)?

Vertical Slices

An alternative approach is to view the graph in *slices*. Let's will consider how to take *vertical* slices. These slices have the advantage of reducing the problem of visualizing multivariable functions to the familiar one of graphing functions of one variable, which we can plot easily in a 2-D plane.

DEFINITION:

Suppose $f : \mathbb{R}^2 \to \mathbb{R},\ \ z = f(x, y)$ is a function of two variables.

The **vertical slice of** f **along the** $x - axis$**, holding** $y = b$ is the function

$$\psi(x) = f(x, b), \quad \text{provided } (x, b) \in U.$$

Similarly, the **vertical slice of** f **along the** $y - axis$**, holding** $x = a$ is the function

$$\phi(y) = f(a, y), \quad \text{provided } (a, y) \in U.$$

Example
Suppose $z = f(x, y) = 2x^2 + 3xy - 4$

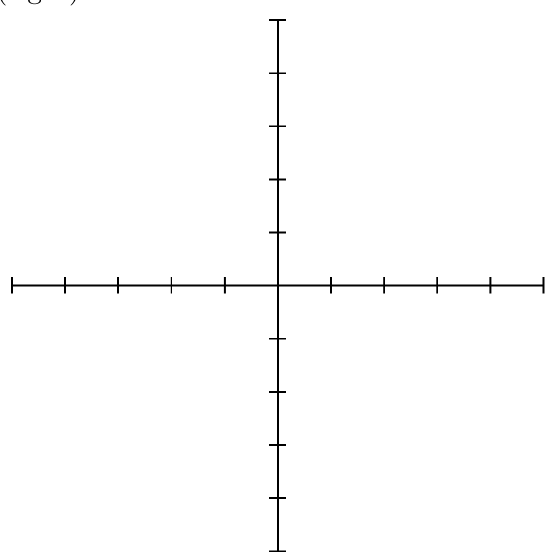
The vertical slice of f parallel to the $x - axis$, holding $y = 3$ is

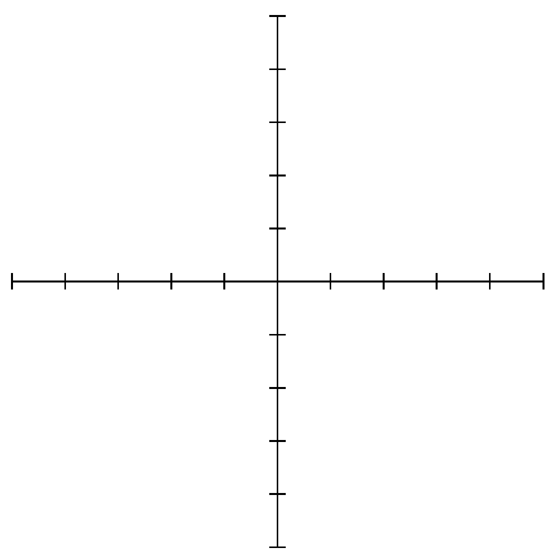
$$z = f(x, 3) = 2x^2 + 3x \cdot 3 - 4 = 2x^2 + 12x - 4 = \psi(x).$$

The vertical slice of f parallel to the y-axis, holding $x = -1$ is $\phi(y) = f(-1, y)$

$$z = f(-1, y) = 2(-1)^2 + 3(-1)y - 4 = -3y - 2 = \phi(y)$$

3. Sketch the graph of $\psi(x)$ versus x below (left) and Sketch the graph of $\phi(y)$ versus y below (right)



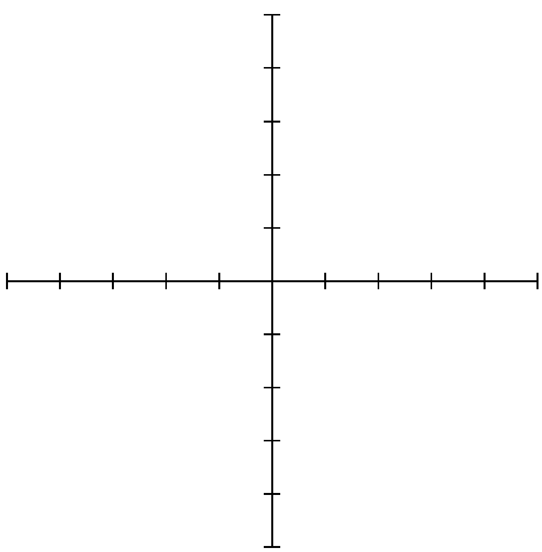


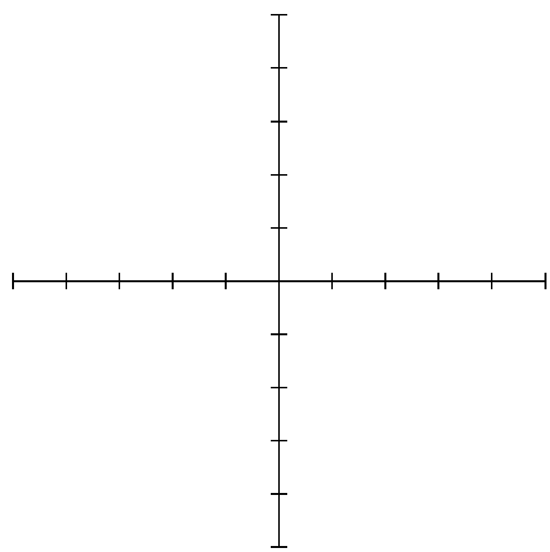
4. Find the equation of the vertical slice of f along the x-axis, holding $y = 0$:

5. Find the vertical slice of f parallel to the y-axis, holding $x = 5/2$:

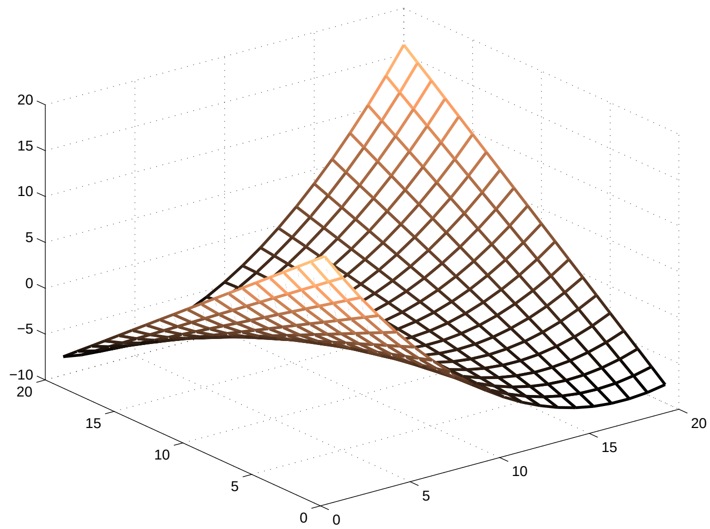
6. What do each of these vertical slices look like for this function $f(x, y) = 2x^2 + 3xy - 4$?

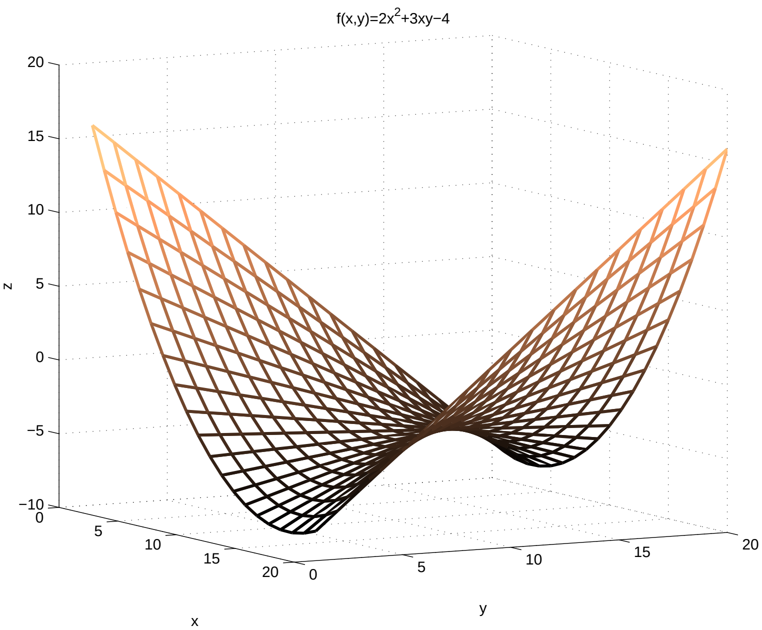
7. Sketch (at least) three vertical slices of $f(x, y) = 2x^2 + 3xy - 4$ where x is held constant on the left axes and (at least) three vertical slices of the same function where y is held constant on the right axes below.

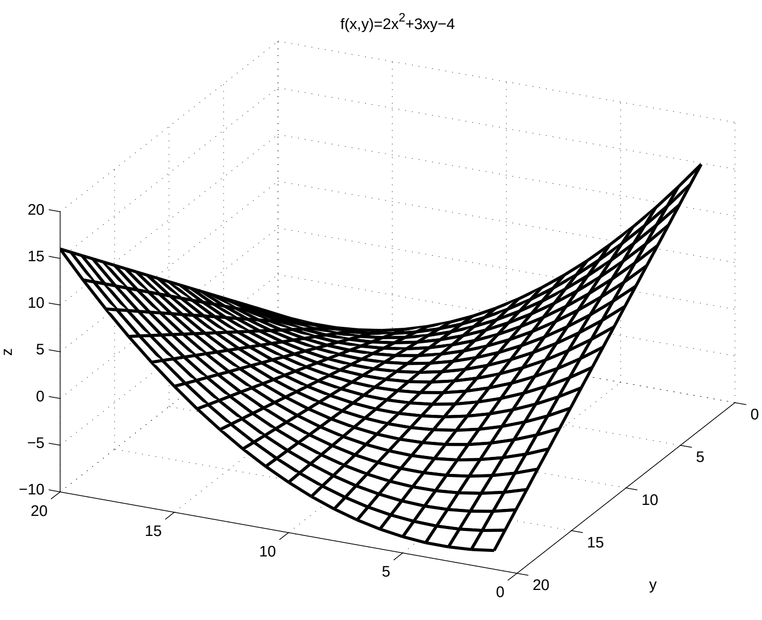




8. Can you use these slices to sketch a picture of the **surface graph** of $f(x, y)$ versus (x, y) below?




f(x,y)=2x^2+3xy−4
z
x
y


f(x,y)=2x^2+3xy−4
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x
y