Application of Differentiation: Optimization

Optimization problems are problems concerned with finding maximum or minimum values of a function. They arise frequently in applications of mathematics, whether in maximizing profit, minimizing energy or cost, or assigning objects to other objects (if you saw the optimization art lecture, this is an example of "assignment" type problems). We are going to discuss optimization throughout the next couple of classes.

Let's begin by reminding ourselves what information the derivatives of functions give us.

Group Work
We are already aware that the derivative gives us qualitative information about the shape of the function. You may recall some of the following concepts. Begin by filling in the necessary information below:

function	**derivative**	**illustration**
increasing		
	negative	
level (flat)		
	undefined	
steep (rising or falling)		
	small (positive or negative)	
straight (horizontal)		
straight (slanted)		

Now we will turn more specifically to the processes of optimization. Maxima and minima occur at special input values called *critical points.* Other words for maxima and minima are *extrema.* The precise definitions follow:
Critical Point: A point $(c, f(c))$ is called a *critical point* of a function $f(x)$ if $f'(c) = 0$ or $f'(c)$ does not exist.

Extremum: The maximum or minimum value a function $f(x)$ outputs on a particular domain is called an **extremum** or extreme value. The plural is extrema.

In other words, possible maxima and minima occur at values where the first derivative equals zero. *Why would this be true?*

Consider the functions $f(x) = x^2$ and $g(x) = -x^2$. Use information you know about these functions in order to fill in the blanks in the following First Derivative Test for extrema.

The <u>First Derivative Test</u> for finding Local Extrema: Let $(c, f(c))$ be a critical point in the domain of $f(x)$:

> If $f'(x)$ is ________________ to the left of c and ________________ to the right of c, then $f(x)$ has a **local minimum** at c.
>
> If $f'(x)$ is ________________ to the left of c and ________________ to the right of c, then $f(x)$ has a **local maximum** at c.

Notice that we used the word *local* to describe the extrema in the first derivative test. How might a local maximum or local minimum compare to a global maximum or global minimum?

Global Minimum and Global Maximum The idea of finding global extrema is very important in word problems.

> $f(x)$ has a *global minimum* at p if all values of $f(x)$ are greater than or equal to $f(p)$.
>
> $f(x)$ has a *global maximum* at p if all values of $f(x)$ are less than or equal to $f(p)$.

Draw a graph of a function that has both local and global extrema:

Not all functions have extrema; of those that do, some will have local extrema, some will have global extrema and others will have both. In fact, any function having a global maximum or minimum has to have a local maximum or minimum since any global extrema are also local extrema. The domain of the function is important in these outcomes.

How to find Local Extreme Values

1. Determine the domain of the function and identify the end points (if any).
2. Find $f'(x)$.
3. Find all roots of $f'(x) = 0$ in the domain, and find where $f'(x)$ does not exist.
4. Use the First Derivative Test to locate any local extrema.

Example: For the function $5x^3 + x^2 - 2x$, find any local maxima and minima on each of the following domains.
(a) $0 \leq x \leq 1$.

(b) $-2 \leq x \leq 0$.

(c) $x \in \mathbb{R}$.

How to find Global Extremes

1. Determine local extrema $(c, f(c))$, as you did above.

2. If the domain of interest is a closed interval, i.e. *Domain* $= [a, b]$, check the end point values $f(a)$ and $f(b)$ and determine which value among $\{f(c), f(a), f(b)\}$ is the largest - this is the global maximum; the smallest is the global minimum.

3. If you do not have a closed interval for the domain, then it is possible to not have global maxima or global minima. For some functions, you can use features of the graph to classify local maxima/minima as global maxima/minima.

Example: For the function $5x^3 + x^2 - 2x$, classify the global maxima and minima on each of the following domains. (Note: *These domains correspond to the domains on which you previously found local maxima and minima.*)

(a) $0 \leq x \leq 1$.

(b) $-2 \leq x \leq 0$.

(c) $x \in \mathbb{R}$.

Recall what the second derivative tells us about the concavity of a function: The graph of a function is **concave up** if the graph stays above the tangent line, and it is **concave down** if the graph stays below the tangent line.

> If $f'' > 0$ on an interval then the graph of $f(x)$ is *concave up* on that interval.
>
> If $f'' < 0$ on an interval then the graph of $f(x)$ is *concave down* on that interval.

We have also seen that a point at which the function changes concavity is called an *inflection* point. *To find an inflection point we set $f''(x) = 0$ or check where f'' Does Not Exist (D.N.E.).*

We can also use second derivative information to gather information about extreme values of a function.
Second Derivative Test for local maxima and minima

Let $(c, f(c))$ be a critical point.

> If $f''(c) > 0$ then $(c, f(c))$ is a local minimum.
>
> If $f''(c) < 0$ then $(c, f(c))$ is a local maximum.
>
> If $f''(c) = 0$ then $(c, f(c))$ is a possible inflection point.

Examples
(a) Sketch $f(x) = \frac{1}{6}x^3 - 2x$ by determining all local maxima and local minima and inflection points.

(b) Sketch $f(x) = x\ln(x)$ on the interval $0 < x < 1$.

Some Optimization Examples
We will deal with some real life problems where it is important to find the maximum and minimum value of some quantity. So in some sense we are trying to optimize the value of some quantity. All the techniques for finding such values make up the field called *optimization*. We shall be able to apply our experience with modeling together with our knowledge of the First Derivative and Second Derivative Tests to help us in our optimization efforts.

Practice

1. A roman window is shaped like a rectangle surrounded by a semi-circle. If the perimeter is L feet, what are the dimensions of the window of maximum area?

2. Based on information provided by the U.S. National Institute on Drug Abuse, the percent, y, of 18- to 25-year-olds who had used hallucinogens between 1974 and 1991 can be modeled by the function

$$y = 0.025x^3 - 0.70x^2 + 4.43x + 16.77$$

where $x = 0$ corresponds to the year 1974. In what year during this period did this type of drug use reach its absolute maximum? Based on this model, what percentage of 18- 25-year-olds had used hallucinogens?