Higher Order Taylor Polynomials

Just about *everything* in this course so far has been based on local approximation of a differentiable function f using a linear function (remember: local linearity!). We can approximate values $f(x)$ near a point $x = a$ linearly using the tangent line. We will refresh our memories to this point in the next example.

Example

Find the equation of the line tangent to the graph of $f(x) = e^{2x}$ at the point $(0, 1)$. Plot this line together with the graph of f. Use this tangent line to approximate $f(1/2)$.

Refresher: First-Degree Taylor Polynomials

The equation you just wrote down for the tangent line should have the form

$$y = P_1(x) = f(a) + f'(a)(x - a).$$

We have seen this before. We know that the tangent line equation can be referred to as the first-degree (or *first-order*) Taylor Polynomial. This has been indicated in the expression above with the notation, $P_1(x)$. The *graph* of the first-order Taylor polynomial for f about the point a is the line tangent to the graph of f at the point $(a, f(a))$. What is true about $P_1(a)$? What is true about $P_1'(a)$?

As useful as the tangent line approximation is, it has some limitations. The biggest problem is that a line doesn't curve–that is, a line has constant slope everywhere while a curve has different slopes at each point. We have spent much time in the past several classes sketching solutions to rate equations. A feature of these solutions we have noticed is their concavity. From this point of view, we can imagine that it would be useful to have an approximating function that takes the local curvature of a function into account. This is possible if our function is twice-differentiable. *Why?*

Second-Order Taylor Polynomials

What does a second-degree polynomial look like in general?

Remember that the first-degree Taylor polynomial, $P_1(x)$ is used to approximate values of $f(x)$ near $(a, f(a))$. We saw that $P_1(x)$ *preserved* $f'(a)$ and $f(a)$. We *want* similar features in the second-degree Taylor polynomial $P_2(x)$.

Say that we know $P_2(x)$ has the general form

$$P_2(x) = c_0 + c_1(x-a) + c_2(x-a)^2,$$

where c_0, c_2, and c_3 are constant coefficients. *Does $P_2(x)$ match your idea of a second-degree polynomial?* In order that $P_2(x)$ resemble $f(x)$ as much as possible for values of x near a, we require that the following three conditions be met:

$$P_2(a) = f(a), \qquad P_2'(a) = f'(a), \qquad P_2''(a) = f''(a).$$

In other words, this new polynomial will also match the function value and the first derivative value at $(a, f(a))$. In addition,it will also match the second derivative value at this point. Now we can use these conditions to find the coefficients c_0, c_1, and c_2 in terms of $f(a)$, $f'(a)$, and $f''(a)$:

These coefficients give us the following expression for a second-degree Taylor polynomial:

Example Find the second-order Taylor polynomial $P_2(x)$ for $f(x) = e^{2x}$ about 0. Add the graph of this polynomial to your previous plot of $f(x)$ and $P_1(x)$. Compare the quality of the approximations to $f(x)$ by $P_1(x)$ and $P_2(x)$ for values of x near $a = 0$.

A note about the Approximation Error

We saw **Taylor's Theorem** before. What did it say?

We can say something similar now. And we will call this new statement the **Extended Taylor Theorem:**

Assume we have all the "nice parts" we need to write a function in the following way:

$$f(a+h) = f(a) + f'(a)h + \frac{1}{2}f''(a)h^2 + E_2(h).$$

The error associated with the second-order Taylor polynomial has the following properties:

$$\lim_{h\to 0} E_2(h) = 0, \quad \lim_{h\to 0} \frac{E_2(h)}{h} = 0, \quad \lim_{h\to 0} \frac{E_2(h)}{h^2} = 0.$$

Note: The error in this theorem (and in the original Taylor's Theorem) is also called the "remainder."

Example For the function $f(x) = e^{2x}$, find an expression for the second-order Taylor error, $E_2(h)$. Show that the $\lim_{h\to 0} E_2(h) = 0, \quad \lim_{h\to 0} \frac{E_2(h)}{h} = 0, \quad \lim_{h\to 0} \frac{E_2(h)}{h^2} = 0.$