Rate Equations, Slope Functions and Concavity

Using the Rate Equation $y' = F(y)$ to Determine y''

So far we have used only first-derivative information in sketching qualitative graphs of solutions $y(t)$ to a rate equation. However, if the slope function $F(y)$ is differentiatiable, then we can use the Chain Rule to get an equation for y''. This information about the second derivative of a solution $y(t)$ will enable us to add more details to our qualitative graphs.

Exercise Obtain an expression for $y''(t)$ given that $y(t)$ satisfies the rate equation $y' = F(y)$.

Now recall what the sign of the second derivative tells you about the graph of a function:

- – **If** $y''(a) < 0$, then $y'(t)$ is decreasing as t increases through a, so the graph of $y(t)$ is ____________________ in a neighborhood of $t = a$.

- – **If** $y''(a) > 0$, then $y'(t)$ is increasing as t increases through a, so the graph of $y(t)$ is ____________________ in a neighborhood of $t = a$.

- – **If** $y''(a) = 0$, then $y''(t)$ **MIGHT** change sign at $t = a$. IF IT DOES CHANGE SIGN, then the concavity of $y(t)$ will change at $t = a$, so the graph of $y(t)$ will pass through an **inflection point** $t = a$.

Finding Inflection Values

DEFINITION: **inflection point**

If a function $y(t)$ passes through an inflection point at $t = a$, and if $y(a) = b$, then b is called an **inflection value** for $y(t)$.

Question

Assume that the rate equation $y' = F(y)$ has a unique solution passing through each point (t_0, y_0), where $F(y_0)$ is defined. If a value y^* is an equilibrium value for $y' = F(y)$, could it also be an inflection value for this rate equation? *Explain.*

Finding Inflection Values

Example

Consider the rate equation $y' = y(1 - y)$. (This is a special case of the logistic equation.) It is not hard to verify that:
The slope function is $F(y) = y(1 - y)$.
The equilibrium values are $y_1^* = 0$ and $y_2^* = 1$.
The corresponding equilibrium solutions are $y_1(t) = 0$ and $y_2(t) = 1$.

- If $y < 0$, then $y' < 0$.
- If $0 < y < 1$, then $y' > 0$.
- If $1 < y$, then $y' < 0$.

The equilibrium value $y_1^* = 0$ is *not* asymptotically stable, but the equilibrium value $y_2^* = 1$ *is* asymptotically stable.

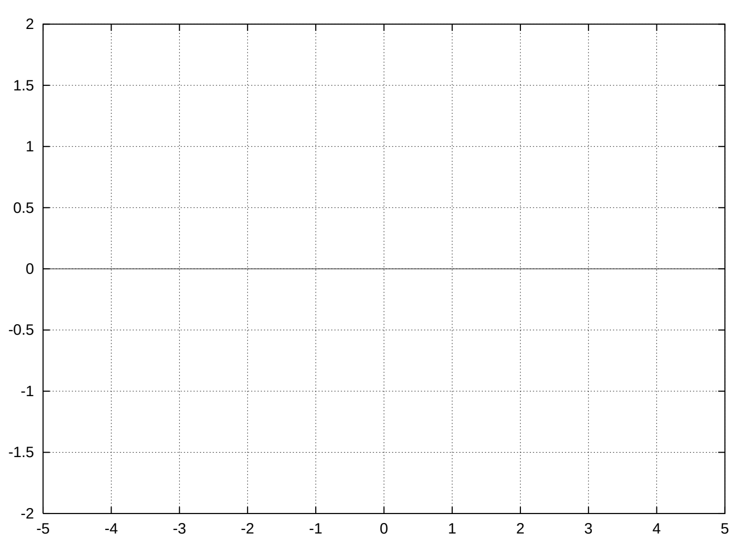
Exercise
Find candidates for inflection values of solutions of $y' = y(1 - y)$

Now check the concavity of $y(t)$ near $\hat{y} = 1/2$:

Observe that $0 = y_1^* < \hat{y} < y_2^* = 1$, and that $y'' = y' \cdot (1 - 2y) = y(1 - y)(1 - 2y)$. Then

$$\begin{aligned} 0 < y < 1/2 \quad &\Longrightarrow\; y'' = y(1-y)(1-2y) > 0 \;\Longrightarrow\; \text{concave up}, \\ 1/2 < y < 1 \quad &\Longrightarrow\; y'' = y(1-y)(1-2y) < 0 \;\Longrightarrow\; \text{concave down}. \end{aligned}$$

We now apply all this information to sketch representative solutions to this rate equation, complete with inflection values on the axes below.



Example

Recall the rate equation $y' = y^2(1-y)$ from *Class 25*. Find the inflection values, if any, for solutions of this rate equation. Use this information, and your previous work, to sketch representative solutions to this rate equation on the axes below.