Stability of Equilibrium Values

Suppose $y(t) = y^*$ is an equilibrium solution to the differential equation

$$y' = F(y).$$

This solution (and the equilibrium value y^*) are said to be **asymptotically stable** if solutions **near** this equilibrium solution tend towards it as time passes.

Note: There are a variety of ways in which an equilibrium solution can fail to be asymptotically stable. Rather than consider all of these, we will simply classify equilibrium solutions (and values) as either asymptotically stable or *not* asymptotically stable.

Example

You have previously seen that the logistic equation $P' = kP \cdot (1 - P/C)$ has equilibrium values 0 and C. Based on your study of this differential equation, determine which, if any, of these equilibrium values are asymptotically stable. *Explain your reasoning.*

Example

Consider the rate equation $y' = (y + 1)(y - 2)$. Find the equilibrium values and solutions for this rate equation. Sketch its slope field. Classify each of its equilibrium values as either asymptotically stable or not asymptotically stable.

Example

Consider the rate equation $y' = \sin(y)$. Confirm that its equilibrium solutions have the form $y(t) = n\pi$, for any integer n.

For which values of n are these equilibrium solutions asymptotically stable? (*Hint: Sketch the slope field.*)

Using Slope Functions for Qualitative Analysis of Rate Equations

You have already learned how to find equilibrium values and equilibrium solutions for a rate equation of the form

$$y' = F(y),$$

and to qualitatively sketch other representative solutions to such an equation. The function $F(y)$ defined by the right-hand side of the rate equation is key to this analysis. This function $F(y)$ is called the **slope function**.

This name is natural because we use the function F to compute the slope of the graph of a solution $y(t)$ to the rate equation. More precisely, when a solution $y(t)$ passes through a point (a, b) with output coordinate $y(a) = b$, then the slope of the line tangent to the graph of $y(t)$ at that point will have the numerical value $F(b)$. Hence the name "slope function" for $F(y)$.

GROUPWORK

Consider the rate equation $y' = y^2(1 - y)$.

1. Identify the slope function $F(y)$.

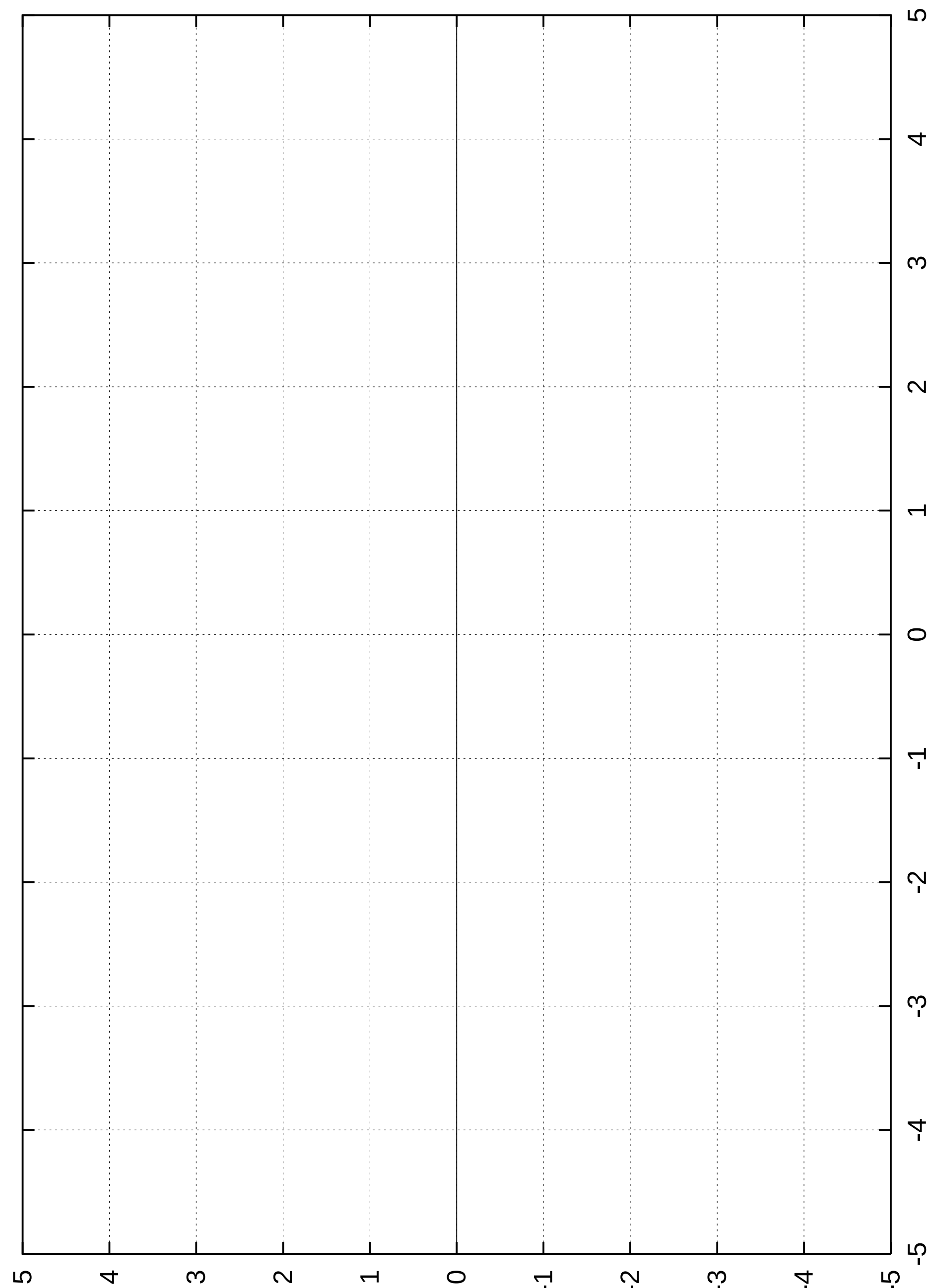
2. Find the equilibrium values, y^*.

3. Plot the corresponding equilibrium solutions, $y(t) = y^*$.

4. Determine the sign of y' between equilibrium values.

5. Sketch representative solutions between equilibrium solutions.

6. Classify the equilibria as asymptotically stable or not.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