Population Models and Qualitative Analysis

Population models are models that track the growth of a population over time; the SIR model we examined some weeks ago is such a model. In the SIR model we tracked three different populations over time: the infected, susceptible and recovered populations. Now we focus on simpler, single population models in order to develop some useful analytical (or investigative) techniques.
Recall that the SIR model equations were based on certain assumptions about the susceptible, infected and recovered populations; if we changed the assumptions, the equations also changed. Similarly, the different population models we explore in the next few classes will be based on the assumptions we make about the population.

Model 1: A relatively simple biological experiment consists of growing unicellular microorganisms, such as bacteria, in a flask and tracking the population change over a period of time. If one is able to observe that over a single unit of time, K new bacteria cells are produced per bacterium. How could we describe this mathematically?

$B(t + \Delta t) \approx B(t) +$

If we consider the limit as $\Delta t \to 0$, we obtain the following differential or rate equation:
$\frac{dB}{dt} =$

The right hand side of the differential (or rate) equation is known as the **slope function** for the population $B(t)$. *Why?*

This type of population growth is known as both **exponential growth** and **Malthus growth**. Sketch the function dB/dt versus B, then use it to sketch a graph of possible solutions $B(t)$ versus t.

Without knowing the solution to the differential equation, we can obtain some information about what the solution does over time. The process of determining properties of the model without "solving" the model is known as **qualitative analysis**. It is an incredible tool–since most models do not have exact solutions (that are known). One of the most important steps in qualitative analysis of population models involves determining special values of the population known as **equilibrium values**. An equilibrium value is a value of the population at which the slope function is zero (or the rate of change of population growth is zero)– hence the name!. Such points are also known as **steady states**, **equilibrium points**, and **singular points**. If a slope function has the form $F(y,t)$, we denote the equilibrium values by y^*.

Does the exponential growth model have any equilibrium values? If so, find the equilibrium values, B^**.*

In doing model analysis, we are also interested in what happens near these values after large amounts of time (i.e. the **asymptotic behavior** of the solutions near the equilibrium values). We will explore this in more detail in the next class.

To revisit some ideas we have touched on before, assume that we know the initial value of the population, $B(0) = B_0$. Then we have a system of equations that should look familiar:

$$\begin{aligned} \frac{dB}{dt} &= KB \\ B(0) &= B_0 \end{aligned}$$

This __________________________ has the following solution:

Note that $\frac{dB}{dt}$ gives us the growth rate of the bacteria. The relative growth rate of the bacteria is given by:

$\frac{1}{B}\frac{dB}{dt} =$

What does the relative growth rate tell us about the population?

How well do you think this model would work in capturing the bacterial growth over time? What are some pros and cons of this population model?

Model 2: Perhaps one of the disadvantages you thought of with regards to the previous model is that it allows the bacterial population to grow without bound. This is hardly ever the case. While small populations do tend to grow in an exponential fashion, larger populations either slow in growth or decline in numbers. *What do you think might cause these effects?*

Competition for resources can have a major effect on populations. How does the following differential (rate) equation account for this kind of effect? (Think about what happens when $P = C$.)

$$\frac{dP}{dt} = KP\left(1 - \frac{P}{C}\right)$$

This type of population growth is known as **logistic growth** and was introduced by Verhulst in 1838. It is applied to many different kinds of populations. Sketch a graph of the function dP/dt versus P and use it to also sketch some solutions $P(t)$ versus t.

Does the logistic growth model have any equilibrium values? If so, find all equilibrium values P^*.

What relationships do you see between the equilibrium values and the two graphs you plotted above?

What do you think happens to solutions near the equilibrium values after long amounts of time?

Assume we know that at time $t = 0$, the population value is given by P_0. Write the initial value problem associated with this population model.

The solution to this IVP is given by:

$$P(t) = \frac{P_0 C}{P_0 + (C - P_0)e^{-Kt}}$$

BONUS HOMEWORK OPPORTUNITY: *On a separate sheet of paper, verify that the solution $P(t)$ is in fact a solution to the initial value problem for the logistic growth model.*

What is the relative growth rate in this case? What does it tell us about the population?

How well do you think this model might work in capturing certain population growth over time? What are some pros and cons of the model?