Implicit Differentiation

Quite often one encounters an equation relating two variables where it is impossible or inconvenient to solve for one of them in terms of the other. An example is

$$y^3 - xy = -6.$$

As x varies in this equation, y must also vary in order to maintain equality. The points (x, y) satisfying this equation lie along some curve in the (x, y) plane.

It is inconvenient to solve the equation $y^3 - xy = -6$ for y in terms of x. In fact, for other equations, it may be impossible to solve for y in terms of x. For example, consider $\cos(y)x^2 - y^3 \sin(x) = 1$. The Chain Rule can nonetheless be used to determine the derivative of $y(x)$, and there are a few immediate uses for this.

Uses of (Implicit) Differentiation

a. Given the derivative of $y(x)$ and one ordered pair (x_0, y_0) satisfying the equation, the Microscope Approximation can be used to approximate values of $y(x)$ for x near x_0. Euler's Method can also be used to approximate $y(x)$ on an interval starting with x_0.

b. The derivative $y'(x)$ may be used to better understand the behavior of the curve $y(x)$, so that, for example, the function $y(x)$ may be optimized by looking for points (x, y) at which $y'(x) = 0$.

To begin, it is very helpful to explicitly write the input variable for $y(x)$. Then differentiate with respect to the input variable, using the Chain Rule. Finally, solve for $y'(x)$. Of course, the derivative will exist only when this expression is defined.

$$\begin{aligned}
(y(x))^3 - xy(x) &= -6 \\
3(y(x))^2 y'(x) - (\,1 \cdot y(x) + x \cdot y'(x)\,) &= 0 \\
y'(x) &= \frac{y(x)}{3y^2(x) - x}
\end{aligned}$$

NOTE: As you become more and more adept at performing these computations, you may begin to omit the explicit notation $y(x)$, writing only y instead. However, with or without the notation, the *understanding* that y is a function of x is crucial.

You may prefer to use the differential notation for derivatives in these problems. In this notation, the example above can be written:

$$\begin{aligned}
y^3 - xy &= -6 \\
3y^2 \frac{dy}{dx} - (1 \cdot y + x\,\frac{dy}{dx}) &= 0 \\
\frac{dy}{dx} &= \frac{y}{3y^2 - x}
\end{aligned}$$

1. Find a value $x = a$ satisfying $y^3 - xy = -6$ when $y = 1$.

2. Find the equation for the tangent line to the curve at the point $(a, 1)$, where a is the value you found above.

3. Use the tangent line or the Microscope approximation to estimate $y(x)$ for x at the value $a + .01$, where a is the value you found in 1. above.

4. Is there any point where the curve has a horizontal tangent? *Hint: Don't forget that points on the curve must satisfy the equation $y^3(x) - xy(x) = -6$.*

5. Is there any point where the curve has a vertical tangent? *Hint: This will occur when the derivative is undefined because the numerator is finite while the denominator is zero.*