The Definition of Derivative

If a function $y = f(x)$ is locally linear at $x = a$, we have informally defined the *slope* or *derivative* of f at $x = a$ to be the slope of the line tangent to the graph of f at the point $(a, f(a))$. That is,

$$f'(a) = \frac{dy}{dx}$$

where the differentials dy and dx are increments taken along the tangent line. Since we rarely actually have the tangent line, this definition is not always useful. The formal definition which we will use instead is based on the *microscope approximation:*

$$f'(a) \approx \frac{\Delta y}{\Delta x}.$$

The increments Δy and Δx are computed from points on the function graph rather than on the tangent line. The values of these increments depend, of course, on the points chosen.

To turn this idea into something useful for actually computing a derivative, we need to recognize two things:

i.) the microscope approximation approaches an equation as $\Delta x \to 0$,

ii.) and in general, there is no "smallest" Δx we can use which will make $\Delta y/\Delta x$ exactly equal to dy/dx.

These two facts together imply the following definition:

Definition: (The Derivative)

The *derivative* of the function $y = f(x)$ at $x = a$ is defined as:

$$f'(a) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}$$

provided this limit exists. In this case, f is said to be *differentiable* at $x = a$.

This limit exists and is equal to both the right and left-hand limits provided these exist and are equal, i.e. provided

$$\lim_{\Delta x \to 0^-} \frac{f(a + \Delta x) - f(a)}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{f(a + \Delta x) - f(a)}{\Delta x}.$$

Example: $f(x) = x^2, \quad a = 1$

We know from experience that f appears to be locally linear at $(a, f(a)) = (1, 1)$, so it makes sense to see if f is really differentiable at $a = 1$:

$$f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1^2}{x - 1}$$

A. To evaluate this limit, we can't simply "plug in" 1 for x. Why not?

B. To continue, we need to simplify this expression algebraically. Factoring is handy here:

$$\frac{x^2 - 1^2}{x - 1} = \frac{(x + 1)(x - 1)}{x - 1} = x + 1, \text{ for } x \neq 1.$$

C. Use this result to finish calculating $f'(1)$.

The Derivative of a Constant Function

1. Suppose that $f(x) = c$, a constant, for all $-\infty < x < +\infty$. Use the definition of the derivative to find $f'(a)$, where a is a particular input value.

The Derivative of a Constant Times a Function

2. Suppose that f is differentiable at a, and that $c \neq 0$ is a constant. Let $g(x) = c \cdot f(x)$. Use the definition of the derivative (and properties of limits) to find $g'(a)$.

The Derivative of a Sum of Functions

3. Suppose that f and g are both differentiable at a. Define $s(x) = f(x) + g(x)$. Use the definition of the derivative (and properties of limits) to find $s'(a)$.

The Derivative of a Linear Function

4. Use the properties already established in this class to find $f'(a)$ if $f(x) = mx + b$, and $m \neq 0$ and b are constants.

Practice with Derivatives: Find the following derivatives.

(a) $f(x) = 3x + 1$

(b) $g(x) = 3x^2 + 1$

(c) $f(t) = \sqrt{3t + 1}$

(d) $h(r) = \dfrac{3}{r + 1}$

Visualizing Derivatives: A function $f(x)$ is plotted below. On the set of axes below, draw the derivative $f'(x)$ as a function of x. On your derivative graph, label the points corresponding to the points **A, B, C, D** and **E** on the original graph.