Differentiability and Linear Approximation

You have already learned that the derivative of a function f at a point a, if it exists, is the slope of the line tangent to the graph of f at the point $(a, f(a))$. In this class you will learn terms that mathematicians use to refer to various aspects of this concept. You will also learn some properties of the error one incurs in using a tangent line to approximate a function graph.

Differentiability at a Point

A function f is said to be *differentiable at a point* a if its derivative $f'(a)$ exists at that point.

As you saw in the last class, a function f which is differentiable at a has a line tangent to its graph at the point $(a, f(a))$. We can also such graphs as being *locally linear* at the point $(a, f(a))$.

Examples

The function $f(x) = x^2 - 1$ is differentiable at $x = 2$. Its derivative there is $f'(2) = 4$, and the line tangent to its graph at the point $(2, f(2)) = (2, 3)$ has the equation $y = 3 + 4 \cdot (x - 2)$.

The function $f(x) = 3x^2$ is differentiable at $x = 1$. Its derivative there is $f'(1) = 6$, and the line tangent to its graph at the point $(1, f(1)) = (1, 3)$ has the equation $y = 3 + 6 \cdot (x - 1)$.

1. Complete the following statement:

   Suppose a function f is differentiable at $x = a$. Its derivative there is $f'(a)$, and the line tangent to its graph at the point $(a, f(a))$ has the equation:

$$y =$$

First-Degree Taylor Polynomial

If a function f is *differentiable at a point* a, then its first degree Taylor polynomial exists and has the formula

$$P_1(x) = f(a) + f'(a)(x - a).$$

Note that the graph of $P_1(x)$ is the line tangent to the graph of f at the point $(a, f(a))$.

Tangent Line Approximation

Suppose f is differentiable at a. Then for values of x near a, the *tangent line approximation* to $f(x)$ is

$$f(x) \approx f(a) + f'(a)(x - a).$$

In this context, the first degree Taylor polynomial $P_1(x) = f(a) + f'(a)(x - a)$ is called the *local linearization* of f about a. Note that the tangent line approximation and the Microscope Approximation are essentially the same thing.

Example of Tangent Line Approximation

Suppose we are "Lost" in the Pacific Ocean and we need to approximate the square root of 29 without a calculator in our desperate attempts to build a raft to get off the island. We can do this using our knowledge of differentiable functions!

1. What's the "easiest" square root we know closest to 29? We'll call this value a.

2. What's the algebraic representation of a function which has as its output value the square root of its input value? We'll call this $f(x)$.

3. Find the equation of the tangent line approximation to this $f(x)$ from 2. at the point $(a, f(a))$ fwhere a is your value from 1.

4. Use this tangent line approximation to estimate the value of $\sqrt{29}$ to 2 decimal places.