Local Linearity and the Microscope Approximation

Local Linearity

A function is said to be *locally linear at a point* if it has a well-defined slope at that point (i.e. if the function has a first derivative at that point). One way to define such a slope is to consider a special line – the line *tangent* to the graph at that point. Then the slope of the graph at that point can be defined as the slope of this *tangent line.*

The tangent line can be approximated visually by zooming in on a point on a function graph with a graphing calculator or a computer program.

1. Suppose a function $y = f(x)$ is locally linear at $x = 2$ and its graph near the point $(2, f(2))$ approaches a line with *slope* $m = -3$ as you zoom in closer and closer. Find a *linear* function $y = g(x)$ whose graph is this line; this linear function gives a good approximation to $f(x)$ for values of x near $x = 2$.

2. Use your linear approximation $g(x)$ to $f(x)$
   for x near 2 to *estimate* $f(2.5)$.
   How is this problem related to Euler's Method?

   The graph of this linear approximation g is the tangent line to the graph of f at the point $(\,2, f(2)\,)$.

One often wishes to distinguish changes in x or y values as you move between points on the tangent line from changes in x or y values as you move between points on the function graph. This is done through a special notation. Increments in x and y in moving between points on the tangent line are called *differentials* and are denoted by dx and dy, respectively.

3. On the tangent line in Problem 1, if x changes by the differential $dx = 2$, what is the corresponding change dy in y? Express the *slope* of the tangent line in terms of the differentials dy and dx.

It is natural to define the *slope* of the function $y = f(x)$ at $x = 2$ to be the slope of the line tangent to the graph of f at $(2, f(2))$. The slope of f at $x = 2$ is denoted $f'(2)$. This is also referred to as the *derivative* of f at $x = 2$ to emphasize that this slope is derived in some fashion from the function f. Thus in our example,

$$\frac{dy}{dx} = f'(2) =$$

The Microscope Approximation

The big problem with defining the slope of a function at a point as the slope of the tangent line at that point is that you still have to find the slope of the tangent line (and the tangent line itself, if it exists, for that matter)! If you zoom in on the graph of a locally linear function (as if you were using a *microscope*) until the graph *appears* like a line, the slope of the tangent line can be *approximated* as the ratio of increments *on the function graph* near the point:

$$\frac{\Delta y}{\Delta x} \approx f'(2).$$

(This ratio is the slope of a *secant* line.) More generally, if $y = f(x)$ is a function which is *locally linear* at $x = a$, if x_1 and x_2 are near $x = a$, and if $y_1 = f(x_1)$ and $y_2 = f(x_2)$, then

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} \approx f'(a).$$

This is called the *microscope approximation* for f at $x = a$. How could this approximation be improved?

4. For what type of function f is the microscope approximation guaranteed to be *exact*? That is, for what type of function is it true that

$$\frac{\Delta y}{\Delta x} = f'(a) \quad ?$$

5. We may rearrange the microscope approximation algebraically as:

$$\Delta y \approx f'(a)\Delta x.$$

What does this have to do with Euler's Method?