Terminology for Initial Value Problems

$$y'(t) = F(t, y(t)), \quad y(t_0) = y_0.$$

The equation $y'(t) = F(t, y(t))$ is called the *rate equation.*

The equation $y(t_0) = y_0$ is called the initial condition.

The right-hand side of the rate equation, viewed as a function of t and y, is called the *slope function, F.*

A function $y(t)$ which satisfies both the rate equation and the initial condition is said to be a *solution* of the initial value problem.

Euler's Method (pronounced "OILER'S METHOD")

To find a *continuous, piecewise linear approximation* $Y(t)$ to the solution $y(t)$ of the *initial value problem*

$$y'(t) = F(t, y(t)), \qquad y(t_0) = y_0,$$

on the interval $t_0 < t \leq t_f$,

i) Decide how many steps N you want to take;
ii) Calculate the stepsize $\Delta t = (t_f - t_0)/N$;
iii) Initialize $Y(t_0) = y_0$.
iii) For $k = 0$ to $N - 1$:
   $t_{k+1} = t_k + \Delta t$,
   $m_k = F(t_k, Y(t_k))$,
   $\Delta Y_k = m_k \cdot \Delta t$,
   $Y(t_{k+1}) = Y(t_k) + \Delta Y_k$,
   Next k.

It is often convenient to record results in a table. In this case, write complete headings to specify the algorithm.

t	$Y(t + \Delta t) = Y(t) + \Delta Y$	slope, $m = F(t, Y(t))$	$\Delta Y = m * \Delta t$

This technique for approximating solutions to initial value problems is called *Euler's Method*. It is named after Leonhard Euler (1707-1783), a great Swiss mathematician who contributed extensively to the development of the Calculus. It is based on the interpretation of the derivative as a slope, and on the *Microscope Approximation* for a differentiable function $y(t)$:

$$\Delta y \approx y'(t)\Delta t$$

Problem 1

Suppose you know that $Y(t)$ is *piecewise-linear* and *continuous*.

Complete the following table to find the values of $Y(1/4)$, $Y(1/2)$, $Y(3/4)$ and $Y(1)$.

Let $\Delta t = 1/4$.

t	$Y(t+\Delta t) = Y(t) + \Delta Y$	slope on $(t, t+\Delta t)$	$\Delta Y = \text{slope} * \Delta t$
0	1	0	
1/4		$-1/4$	
1/2		$-1/2$	
3/4		$-3/4$	
1			

As this example shows, a piecewise-linear and continuous function is completely determined by its *initial output value* at an *initial input value* together with the *slope* used on each interval over which the function is linear.

Problem 2

This is like Problem 1, except that the slope *on the interval* $(t, t+\Delta t)$ is given by the formula *slope* $= -Y(t)$.

Complete the following table to find the values of $Y(1/4)$, $Y(1/2)$, $Y(3/4)$ and $Y(1)$.

Let $\Delta t = 1/4$.

t	$Y(t+\Delta t) = Y(t) + \Delta Y$	slope on $(t, t+\Delta t)$ m $= -Y(t)$	$\Delta Y = m * \Delta t$
0	1		
1/4			
1/2			
3/4			
1			

Example

The table you completed as "Problem 2" above gives Euler's Method, with a stepsize of $\Delta t = 1/4$, for finding a piecewise linear approximation to the solution $y(t)$ of the initial value problem

$$y(0) = 1, \qquad y'(t) = -y(t), \qquad 0 < t < 1.$$

Study that table carefully, paying particular attention to notation. Then set up and complete a similar table for Euler's Method for the same initial value problem, this time using a stepsize of $\Delta t = 1/8$.

The *exact* solution to this initial value problem happens to be $y(t) = e^{-t}$. Relative to this exact solution, how does the approximation produced by Euler's Method change as the stepsize is decreased?

To Check Your Understanding

a) What is meant by "a solution to an initial value problem?"

b) How is Euler's Method related to the point-slope form of the equation for a line?

c) Euler's Method gives us a sequence of points of the form $(t_k, Y(t_k))$, $k = 0, 1, 2, \ldots, N$. How do you get from this sequence of points to a continuous, piecewise-linear approximation?

d) Why does Euler's Method generally produce only an approximation to the solution and not the exact solution?

e) What considerations enter into deciding the number of steps to take (or equivalently, the stepsize to use)?