Introduction to Mathematical Modeling

A model is not an exact representation of a real phenomenon.

Purpose: Why construct a model?

Assumptions: What is being simplified and omitted?

Validation: How do you know if the model is satisfactory?

Modeling with Rates

It is often easier to construct a model involving *rates of change* of a variable than to model the variable directly. This is the basis of many applications of differential calculus to science.

Proportionality If the rate of growth of a population P' is *directly proportional* to the current population P, we express this as

$$P' \propto P$$

which, written as an equation is

$$P' = kP$$

where k is known as the *proportionality constant* of the relationship between P' and P.

Constant Rate If a car travels at a constant *rate* of 55 mph, what is the *slope* of the graph of the distance it travels as a function of time?

Slope and Rate If it is possible to approximate the graph of a function locally by a tangent line, we can similarly interpret the *rate of change* of the function at a given point as the *slope of the line tangent* to its graph at that point.

Newton's Law of Cooling

Consider a cup of hot water cooling to *ambient temperature*, A. (This is the temperature of the surrounding environment. We assume ambient temperature remains constant. In our experiment, the surrounding environment consists of the ice bath and the air above it, with an average temperature of about 3 degrees Celsius.)

Let $H(t)$ denote the temperature of the water at time t.

Let $H'(t)$ denote the rate of change of H at time t.

Purpose of the Model:

Assumptions:

Constructing the Model:

Convention

A rate is said to be *positive* if it denotes an *increase*, and *negative* if it denotes a *decrease*. A rate of *zero* denotes *no change*.

If $H(t) - A > 0$, is $H'(t)$ positive, negative, or zero?

If $H(t) - A = 0$, is $H'(t)$ positive, negative, or zero?

If $H(t) - A < 0$, is $H'(t)$ positive, negative, or zero?

Propose a *simple* model which relates $H'(t)$ and $H(t) - A$.

Another Example
Suppose the coffee is at 180 degrees and the ambient temperature is 70 degrees, the coffee is cooling at a rate of 9 degrees per minute.

1. In this case, what is the value of K, the constant of proportionality?

2. Do you expect this value to be positive or negative?

3. At what rate is the coffee's temperature changing after it has cooled down to 135 degrees?

4. If the temperature of the coffee is initially 180 degrees, and it is cooling at 9 degrees per minute *estimate* its temperature H after 1 minute, i.e. $H(1)$.

5. If the temperature of the coffee is initially 180 degrees, *estimate* its temperature H after 5 minutes, i.e. $H(5)$.

6. If the temperature of the coffee is initially 180 degrees, *estimate* its temperature H after 10, i.e. $H(10)$ and 20 minutes, i.e. $H(20)$.

7. If the temperature of the coffee is initially A degrees, *estimate* its temperature $H(t)$ after t minutes.

How confident are you of your estimates of the coffee temperature? Do they "make sense"?

Our answers for $H(1)$, $H(5)$, $H(10)$, $H(20)$ et cetera are **estimates** because the rate at which the coffee cools down **changes** as it's cooling down. In making our estimates, we are assuming that the rate of cooling down is constant over certain time intervals (1 minute, 5 minutes, or 10 minutes.) Unfortunately it is not!

(What time interval would give us the most accurate estimate?)
Which of the following gives a better estimate of the temperature after 10 minutes:

(a) We assume $H'(t)$ is constant for the entire ten minutes. Then $H(10)$ is obtained as we did above.

OR

(b) We assume that $H'(t)$ is constant for the first 5 minutes and calculate $H(5)$ as above. Then recalculate the rate of change, $H'(5)$, based on the new temperature and determine the change in temperature ΔH over the next 5 minutes based on the new rate of change. Add this change to the estimate for $H(5)$ to get $H(10)$.

This idea of recalculating the rate of change frequently (every 5 minutes in this case) forms the basis of **Euler's Method** — a technique to approximate solutions to initial value problems.