Data and Population Models

In this lab, we will relate some population data to the populations models we have seen in class.

The following table gives U.S. population data at various times between 1790 and 1860:

Table 1: US population data between 1790 and 1860

Year	1790	1800	1810	1820	1830	1840	1850	1860
Population (millions)	3.9	5.3	7.2	9.6	12.9	17.1	23.2	31.4

We are interested in determining whether this data can be fit to a model we have seen in class. This lab will guide you through that process. We approach our problem by first estimating the relative growth rates for the US population over each time interval. What is the relative growth rate?

Hint: You may find it useful to to first estimate the rate of growth of the population between each interval. Store your estimates in the table below:

Table 2: My Estimates

	Year	1790	1800	1810	1820	1830	1840	1850	1860
growth rate	$\frac{dP}{dt}$								
relative growth rate									

Start by determining whether the exponential growth model adequately describes the US population data over the time period 1790-1860. Write down the IVP for the exponential growth model as well as its solution in the space below:

In lab we will focus on using the solution to the IVP to determine whether the model fits the data. There is another way to determine this; what is it? (*Hint: What would you have to do if we didn't know the exact solution?*) Start by determining our value for k. (*Hint: Solve for k in the rate equation. Upon doing so, what familiar growth related expression do you see?*

Next, account for the initial condition. What should it be? (*Hint: Think about what the initial year is in our situation. How is this different from the initial condition in the IVP above?*)

How should the solution form you wrote down above be modified in order to account for the initial data? *Check with me before proceeding!*

Now use the correct solution form for your particular IVP to predict the population value in 1860.

How does it compare with the actual value given by the data?

More Parameter Estimation You estimated our growth rate parameter k from estimates of the relative growth rate. Estimating parameters for model equations is a tricky process and is approached in a variety of ways. Using your particular solution, you can estimate k using another method. Once again, write down the solution form you obtained above.

Now, take the natural log of both sides to obtain an expression for $y = \ln P(t)$. (*Hint: Remember to use the correct properties of logs.*)

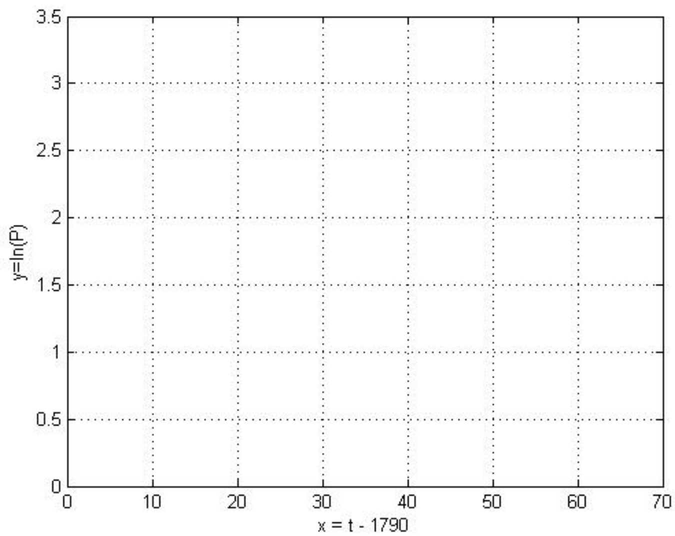
Set $x = t - 1790$ and substitute this into your expression. What kind of function is y in terms of x?

What is the y-intercept of this line? What is the slope of this line?

Complete the table below for x and y. Then plot the points on the graph below. Draw the line that you believe is the "best" fit to all of the points and estimate its slope. *This slope provides an estimate for* ______________.

Table 3: x and y estimates

t	1790	1800	1820	1840	1860
x $(t - 1790)$					
P(t)	3.9	5.3	9.6	17.1	31.4
y $(\ln P(t))$					



What was your new value for k? How did it compare to the old value of k? Which provides a better estimate for the population in 1860? Would there be a better value of k?

More Data! The US population has changed since 1860; hence we should be able to find data on the US population size since then. Assume that the exponential growth model holds through 1940. What population value does the model predict at this time?

More data does exist on the US population size since 1860. Some of this data is presented in table below. How does the predicted population value you just computed compare to the actual population value?

Table 4: US population data between 1860 and 1940

Year	1860	1870	1880	1890	1900	1910	1920	1930	1940
Population (millions)	31.4	38.6	50.2	62.9	76.0	92.0	105.7	122.8	131.7

You probably found that the estimate for the population value does not match the actual population data very well. So, now it is time to try to improve or change our modeling approach. Start once again, by estimating the rate of the change of the population and the per capita growth rate of the population over each time interval.

Table 5: Estimates for $\frac{dP}{dt}$ and $\frac{1}{P}\frac{dP}{dt}$ for the US population between 1860 and 1940

Year	1860	1870	1880	1890	1900	1910	1920	1930	1940
$\frac{dP}{dt}$									
$\frac{1}{P}\frac{dP}{dt}$									

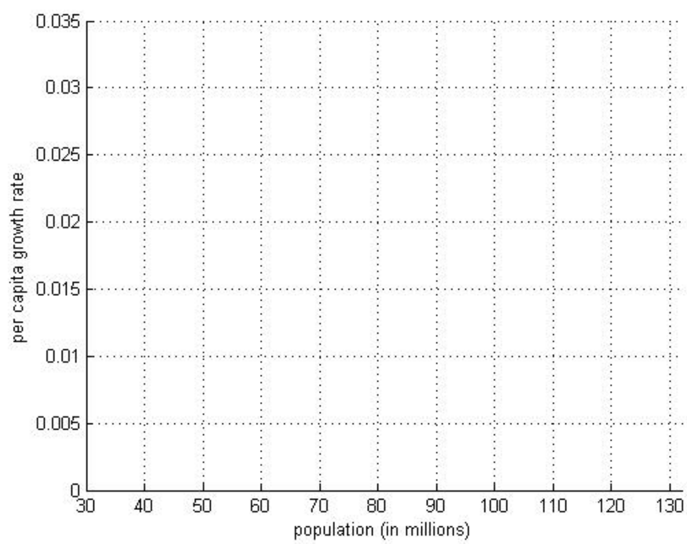
What is happening to the per capita growth rate during this time period?

After 1860 the per capita growth rate of the U.S. population began to decrease. Because of this, the assumptions of the exponential model are not valid for this period and a new model is needed. What is the next simplest model we have seen that will allow for the per capita growth rate to decrease?

Write down the rate equation as we saw it in class:

We can also write this equation in the form $\frac{dP}{dt} = kP - aP^2)$. We will use this form for simplicity in the lab. How are k and a related to the parameters in the form of the model we saw in class? Solve the new form for the per capita growth rate.

According to these calculations, if this particular model holds, then the per capita rate should be a *linear* function of the population rather than simply a constant. Plot a graph of the relative growth rate versus population size on the graph below. According to the graph, is it justified to use this model over this time period?



How does your graph compare to a graph with a P-intercept of 0.0318 and slope $-.000170$? (Hopefully, relatively well!) Use these values to determine the parameters in this model's rate equation.

We can check how the model compares to the data, especially since we know the solution to an IVP of this type. What is the solution?

What is our IVP?

Does the solution above need to be "adjusted" at all to satisfy our IVP? If so, what is that particular modification?

Using the correct solution form, find the model value for the population in 1940. How does this compare to the data reported? How does it compare to the estimate given by the exponential model?

Use this new model to estimate the population size back in 1760. How does this value compare to the actual value?

Use the model to make an estimate for the population in the year 2000. Find a data point for this value (for example, on the internet) and compare. Make sure to remember where you found your data!

Lab Write Up Instructions

In this lab you have walked through how one might choose models for certain populations. You have had some experience using data in order to decide how good a model is as well as how one might go about finding the relevant parameters. Please provide a well-written, proof-read 3 page technical paper discussing the two different population models in reference to the US population between 1790 and 1940. You should indicate whether you think one model or the other works better over this time period, or if one or the other works better at certain time; explain why you think this. Speculate on the reasons the models work or don't work. You are able to have any opinion (i.e. there is no "right" answer as to how well you think they work); however, you should justify all your opinions. So give explanations and evidence. What modifications, if any, might provide a better fit. Relate your answers to the process of parameter estimation. Would different parameter fitting techniques make the models work better? Speculate on whether estimating these particular parameters is easier or more difficult than you might encounter with other types of models. Think about whether having the exact form of the solution helped. How would you have estimated the parameters without them?

Remember that you want your paper to be self-contained, so that anyone could read it without having done the lab or attended our classes. Provide relevant definitions and explanations. Include figures and refer to them by names and provide them with labels. Remember that spell-checkers will not tell you if you have a word like "form" where you should have "from" or "than" when you should have "then." Conclusion: PROOFREAD! Better yet, get a non-Math114 student to read your paper and provide you with feedback on its "comprehensibility." In other words, provide **CONTEXT:** to educate the reader; **EVIDENCE:** to bolster your observations and conclusions; and **DETAILS:** sufficient for replicability.

This report will be due 2 weeks from today, on **Monday, November 14 or Tuesday, November 15**.