Lab 3: Limits, Continuity, and Differentiability

Objectives:

1. To become familiar with the program **Derive**.
2. To examine the concepts of limit, and continuity.
3. To determine differentiability of a function at a point by taking limits of difference quotients.
4. To posit a relationship between the differntiability and continuity of a function at a point

Introduction: Using *Derive*

The lab this week will introduce you to another computing resource which we will use in this course. *Derive* is avaliable on the network under the *Mathematics* icon. It is a menu-driven package which specializes in symbolic computation. "Menu-driven" means that you work with the package by selecting the features you want from a menu rather than by writing or modifying a program. "Symbolic computation" refers to computing with symbols as well as with numerical values. For example, *Derive* is able to factor $x^2 - 1$ as $(x + 1)(x - 1)$. Excel and the TI-83 are unable to do this.

Derive also has rather nice features for exploring function graphs. This is what we will use it for today. To begin, click on the *Derive* icon under *Mathematics*. A screen will appear with a list of menu options and buttons at the top. This particular screen is called the "Algebra" window in *Derive* because this window will be used to author and modify algebraic and other expressions. The first thing we will be doing is looking at the graph of the function

$$p(x) = \sin(x)/x$$

for values of x near 0.

1. What is the domain of $p(x) = \sin(x)/x$?

Use the mouse to select an option. If you leave a mouse on a menu button for a moment, a message will appear telling you what that button does. Now select **Author**.

Anytime you want to enter an expression in *Derive*, you select **Author** in the algebra window, then **Expression**. When you do so, a small window opens up in which you can enter your expression.

Type

$$\sin(x)/x$$

in the authoring window, then < *Enter* > it. The authoring window will disappear and your expression will appear in the algebra window.

To plot the function whose rule is given by an expression, make sure the expression is *highlighted* in the algebra window, then click on the second button from the right in the toolbar. Do this now.

The screen is now replaced with a "graphics" window. This window will have a pair of axes marked with tickmarks and its own menu at the top. Now select and enter **Plot** from this menu. The graph of this function should appear.

Note that although $p(0)$ is undefined (why?), *Derive* does not seem to show it on the graph.

§1 Exploring a Graph with *Derive*

Derive has several features which allow you to explore graphs. First, notice the cross-hairs. They can be controlled with either the mouse or the "arrow" keys. At the bottom of the screen you will see the x- and y-coordinates changing as you move the cross-hairs around.

Now examine the bottom of the screen more closely. The spacing between the tickmarks on the x and y axes will appear as **Scale** in the format *x-scale*: *y-scale*. What are these values now?

There are several features of the graphics window menu which we will also be using. Select **Set**, then **Center**. Type 0 for the *Horizontal* coordinate and 1 for the *Vertical* coordinate, then $<$ *Enter* $>$. Describe what happens. (Also note the *Center* box at the bottom of the screen.)

The other feature we will be using is **Zoom**. Various sorts of zooming are possible. These are performed by the buttons at the right side of the menu bar with little arrows on them. Find and select the button which *zooms in on both axes*. Describe what happens. Pay particular attention to the values for the x and y scales.

You now know the basics of working with *Derive*. During the rest of the lab, we will be using the following sequence of operations to focus on certain points of the graph of a function.

Move the cross-hairs to the point of interest.

Center *the window on that point.*

Zoom *in on the center of the window.*

Try zooming in and out on various points just to get the hang of this sequence of operations.

§2 Limits

Before continuing with the function $p(x) = \sin(x)/x$, let's first examine more closely the concept of "limits" by computing $\lim_{x\to 0} \sin(x)$. We do this in two steps:

(i) Find the limit as x approaches zero from the **right**: $\lim_{x\to 0^+} \sin(x)$;

(ii) Find the limit as x approaches zero from the **left**: $\lim_{x\to 0^-} \sin(x)$.

Use the table below to perform these two steps.

x	$\sin(x)$	x	$\sin(x)$
1		-1	
0.1		-0.1	
0.01		-0.01	
0.001		-0.001	
0.0001		-0.0001	

(i) $\lim_{x\to 0^+} \sin(x) = ?$

(ii) $\lim_{x\to 0^-} \sin(x) = ?$

What does this say about $\lim_{x\to 0} \sin(x)$?

§3 Continuity

2. Recall we are working with the function $p(x) = \sin(x)/x$. Use *Derive* and/or your calculator to estimate $\lim_{x\to 0} p(x)$. Look at $p(x)$ for the following sequences of x values approaching $x = 0$ from the left and from the right.

x	$p(x)$	x	$p(x)$
-1		1	
-0.1		0.1	
-0.01		0.01	
-0.001		0.001	
-0.0001		0.0001	

Do you think you would get the same results for other sequences of x values approaching 0 from below or from above? *Zooming in* on the graph with *Derive* may help you decide. Based on your conclusion, determine the following limit or explain why it does not exist.

$$\lim_{x\to 0} p(x) =$$

Informally, a function is *continuous* at a point if its graph is unbroken there. Another way to think of this is: a function is continuous if you can draw its graph without lifting the pencil off the paper. This idea can also be expressed in terms of limits.

Definition: A function $g(x)$ is *continuous* at a if $\lim_{x\to a} g(x) = g(a)$.

3. Based on this definition, is $p(x) = \sin(x)/x$ continuous at $x = 0$? *Hint: What is the value of $p(x)$ at $x = 0$?*

4. Complete the following definition so that the function $q(x)$ *is* continuous at $x = 0$:

$$q(x) = \begin{cases} \frac{\sin(x)}{x} & \text{if } x \neq 0 \\ & \text{if } x = 0 \end{cases}$$

§4 The Derivative of the Absolute Value Function

The absolute value function is denoted by "abs(x)" in *Derive*. Use *Derive* and the methods you have just learned to first obtain a graph of $y = f(x) = |x|$, and then estimate the slope of the graph at $x = 0$.

Δx	$\frac{f(0+\Delta x)-f(0)}{\Delta x}$	$\frac{f(0)-f(0-\Delta x)}{\Delta x}$	$\frac{fr(0+\Delta x)-f(0)-\Delta x\|}{2\Delta x}$
0.1			
0.01			
0.001			

5. Do the results of these three methods agree?

6. Do you think it is possible to define the slope of the graph of the absolute value function at $x = 0$? Why or why not?

Write-Up
This worksheet and discussions you have with your lab partners are the basis for your report. You need **hand in only one report per group**, but everyone is expected to contribute. Everyone receives the same grade. You should **include a cover sheet with your names and signatures**. The write-up should be a short (1 or 2 pages) essay which synthesizes the ideas underlying the calculations performed. You may wish to include data or graphical sketches which clarify points made in your report. Your well-written, grammatical essay should answer the questions on the lab worksheet and, in particular, discuss the following: Taking limits from the left and from the right, and comparing them; How to determine whether a given function is continuous at a given point; using the left and right limit of the difference quotient to find the derivative at a point and the relationship between the differentiability and continuity of a function at a point. **Your Lab Report is due one week from today, October 10 or 11, 2005.**