NAMES: ______________________________

Lab 7: Newton's Method

Objectives:

1. To implement Newton's Method numerically to find the root of a function
2. To develop an understanding of Newton's Method computationally, analytically and graphically
3. To be introduced to and appreciate the concept of iteration, recursion and convergence
4. To determine and understand situation where Newton's Method fails

Warm-up

Definition: The value r is a **root** of the function g if $g(r) = 0$.

Find the roots of the function $g(x) = (x-3)^4(x+2)$.

Consider the function $f(x) = x^3 - 2$. Find the equation of the tangent line to this function at $x = 1.5$.

What is the root of this tangent line, i.e. what are the coordinates of the point where this tangent line crosses the x-axis?

Derivation of Newton's Method

Consider a general function $h(t)$. We would like to get an approximation of a root, t^* of this function, and although we do not know the exact value of t^*, we can give a rough first approximation of $t = t_0$ near to $t = t^*$.

Consider the line tangent to $h(t)$ at $t = t_0$.

1. What is the slope of this line? (While we cannot write it down numerically, we do have notation that describes this slope.)

2. Name a point that this tangent line must pass through. (Give both coordinates.)

3. Using the slope and point of the tangent line, determine the equation of the tangent line (in the slope-intercept form $y = mt + b$).

$$y = \underline{\hspace{3cm}}\ t + [\underline{\hspace{6cm}}].$$

4. Now find the root $t = t_1$ of this tangent line, i.e., where the line crosses the t-axis. Simplify the expression for t_1 as much as possible. (Notice that your answer t_1 is dependent upon t_0, i.e. is a function of t_0.)

5. Repeat the process, finding an expression for the root $t = t_2$ of the equation of the line tangent to h at the point $t = t_1$. (Notice also that your answer t_2 is dependent upon t_1, i.e. is a function of t_1.)

   A **recursive algorithm** is one in which your next estimate t_{n+1} is produced using information from your current estimate t_n.

6. Write a general recursive formula for the root $t = t_{n+1}$ of the equation of the line tangent to the curve $h(t)$ at the point $t = t_n$. Check this formula with other lab teams and/or with the instructor.

Testing our recursive formula

We will use the recursive formula on a specific case. Let $h(t) = t^3 - 2$.

1. Find $h'(t)$.

2. Start with $t_0 = 1.5$. Calculate $h(t_0)$ and $h'(t_0)$. Use the recursive formula to find t_1, the root of the line tangent to the graph of h at t_0.

3. Calculate $h(t_1)$ and compare the value with $h(t_0)$. Is t_0 or t_1 closer to the root of h? How do you know?

4. On the space below, draw the graph of h illustrating the determination of t_1.

5. Repeat (or **iterate**) the Newton recursive process and fill in the table below:

Step	t_n	$h(t_n)$	$h'(t_n)$
0	1.5		
1			
2			

Has the sequence of estimates t_n converged? How confident are you that you know the limit $\lim_{n\to\infty} t_n$? How accurately? What could you do to obtain the limit of this sequence accurately?

Computer Iterations

Now that we have the hang of Newton's method by hand, we will get the computer to do the iterative process for us. Open the Excel spreadsheet `newton.xls` (in the `S:\Math Courses\Math114` directory) where the basic format is set up for you. Carefully fill in the formulae for the table, and then, as a check, compare your results above with those in the first few lines of spreadsheet.

1. Suppose we want to find the solutions of the equation $x^3 + 2x^2 - 10x = 20$. Define a function $f(x)$ such that the solutions of $x^3 + 2x^2 - 10x = 20$ are identical to the solutions of $f(x) = 0$, i.e., the roots of f. Once you have the definition of f, adjust the spreadsheet as needed and find all the roots. How are you sure that you have all the roots?

2. Now investigate the problem $x^3 + 3x^2 - 2x - 4 = 0$. Suggestion: start with $x_0 = 0$. What happens? Why? Has Newton's method failed us, or were we just unlucky to pick this starting point? Draw a sketch of the graph of your function and illustrate what is going on with the recursive process.

3. Now consider the equation $\cos(x) = x$. Define f appropriately in the spreadsheet and start with $x_0 = 16$. What is happening? Again, draw a sketch of the graph of your function and illustrate what is going on.

Assignment

Each team will turn in one neat copy of this lab with the signature of all the group members on the coversheet. You may need to have attachments which contain sketches or Excel output. This is due in lab on **Monday November 14 or Tuesday November 15**.