HOMEWORK 6: SOLUTIONS

2.4: 6, 19, 20, 25, 26, 45, 47

0.7: 17, 26,

2.7: 1, 31, 38,

2.8: 1, 2, 12, 20.

2.4: 6. $f(x) = (x^3 - 2x^2 + 5)(x^4 - 3x^2 + 2)$

Using the product rule in combination with the power rule:

$$f'(x) = (x^3 - 2x^2 + 5)(4x^3 - 6x) + (3x^2 - 4x)(x^4 - 3x^2 + 2)$$

Simplifying:

$$= 4x^6 - 8x^5 + 20x^3 - 6x^4 + 12x^3 - 30x$$
$$+ 3x^6 - 9x^4 + 6x^2 - 4x^5 + 12x^3 - 8x$$

$$\boxed{f'(x) = 7x^6 - 12x^5 - 15x^4 + 44x^3 + 6x^2 - 38x}$$

19. $f(x) = \dfrac{x^2 + 3x - 2}{\sqrt{x}} = \dfrac{x^2 + 3x - 2}{x^{1/2}}$

There are 2 ways to take this derivative You can use the quotient rule (i) or simplify (ii):

(i) Using the quotient and power rules:

$$f'(x) = \frac{(x^{1/2})(2x + 3) - (x^2 + 3x - 2)(\frac{1}{2}x^{-1/2})}{(x^{1/2})^2}$$

$$= \frac{2x^{3/2} + 3x^{1/2} - \left(\frac{1}{2}x^{3/2} + \frac{3}{2}x^{1/2} - x^{-1/2}\right)}{x}$$

$$= \frac{2x^{3/2} + 3x^{1/2} - \frac{1}{2}x^{3/2} - \frac{3}{2}x^{1/2} + x^{-1/2}}{x} \cdot \frac{x^{1/2}}{x^{1/2}}$$

$$= \frac{2x^2 + 3x - \frac{1}{2}x^2 - \frac{3}{2}x + 1}{x^{3/2}}$$

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Continuing from previous page:

$$= \frac{\frac{3}{2}x^2 + \frac{3}{2}x + 1}{x^{3/2}} = \frac{3}{2}\left(\frac{x^2}{x^{3/2}}\right) + \frac{3}{2}\left(\frac{x}{x^{3/2}}\right) + \frac{1}{x^{3/2}}$$

$$\boxed{f'(x) = \frac{3}{2}x^{1/2} + \frac{3}{2}x^{-1/2} + x^{-3/2}}$$

(ii) Simplifying first:

$$\frac{x^2 + 3x - 2}{x^{1/2}} = \frac{x^2}{x^{1/2}} + \frac{3x}{x^{1/2}} - \frac{2}{x^{1/2}} = x^{\frac{3}{2}} + 3x^{1/2} - 2x^{-1/2}$$

Now using the power rule:

$$f'(x) = \frac{3}{2}x^{1/2} + 3 \cdot \frac{1}{2}x^{-1/2} - 2\left(-\frac{1}{2}\right)x^{-3/2}$$

$$\boxed{f'(x) = \frac{3}{2}x^{1/2} + \frac{3}{2}x^{-1/2} + x^{-3/2}}$$

20. $f(x) = \dfrac{2x}{x^2+1}$

Using the quotient rule and power rule:

$$f'(x) = \frac{(x^2+1)(2) - (2x)(2x)}{(x^2+1)^2} = \frac{2x^2 + 2 - 4x^2}{(x^2+1)^2} = \frac{-2x^2+2}{(x^2+1)^2}$$

$$\boxed{f'(x) = \frac{-2x^2+2}{(x^2+1)^2}}$$

25. We have a "big" function: $\lambda(x) = f(x)g(x)h(x)$

Let's call $f(x)g(x) = p(x)$. Then $\lambda(x) = p(x) \cdot h(x)$.

According to the product rule:

$$\lambda'(x) = p(x) \cdot h'(x) + p'(x) \cdot h(x)$$

But, also according to the product rule (and the definition $p(x) = f(x)g(x)$):

$$p'(x) = f(x)g'(x) + f'(x)g(x)$$

So, substituting:

$$\lambda'(x) = f(x)g(x)h'(x) + [f(x)g'(x) + f'(x)g(x)]\,h(x)$$

So: $\boxed{\lambda'(x) = f(x)g(x)h'(x) + f(x)g'(x)h(x) + f'(x)g(x)h(x).}$

In general,

$$[f_1(x)\, f_2(x)\, f_3(x) \cdots f_n(x)]'$$

$$= f_1'(x) \cdot f_2(x) f_3(x) \cdots f_n(x)$$
$$+ f_1(x) f_2'(x) f_3(x) \cdots f_n(x)$$
$$+ \cdots$$
$$+ f_1(x) f_2(x) f_3(x) \cdots f_n'(x)$$

That is, the derivative is the sum of n terms in each of which the nth function is differentiated.

26. $(g(x))^{-1} = \frac{-1}{g(x)}$

So, according to the quotient rule:

$$\left[\frac{-1}{g(x)}\right]' = \frac{g(x)\cdot 0 + g'(x)(-1)}{(g(x))^2} = \frac{-g'(x)}{(g(x))^2} = -g'(x)(g(x))^{-2}$$

According to the product rule:

$$\left[f(x)(g(x))^{-1}\right]' = f(x)\{(g(x))^{-1}\}' + f'(x)(g(x))^{-1}$$

Then using the first result and substituting it in:

$$= f(x)\{-g'(x)(g(x))^{-2}\} + f'(x)(g(x))^{-1}$$

$$\boxed{\left[f(x)g(x)^{-1}\right]' = \frac{-f(x)g'(x)}{(g(x))^2} + \frac{f'(x)}{g(x)}}$$

45. $F(x) = f(x)g(x)$

$F'(x) = f'(x)g(x) + f(x)g'(x)$, according to the product rule.

Using the product rule again (twice now!), we obtain:

$$F''(x) = f'(x)g'(x) + f''(x)g(x) + f(x)g''(x) + f'(x)g'(x)$$

$$\boxed{F''(x) = f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x)}$$

We know $(a+b)^2 = a^2 + 2ab + b^2$. If, in this form, the power represents the derivative (ie $1 \Rightarrow$ first derivative and $2 \Rightarrow$ second derivative) and $a = f(x)$, $b = g(x)$ then $F''(x)$ matches the form $(a+b)^2$.

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Using the product rule three times (once per term) on $F''(x)$:

$$F'''(x) = f''(x)g'(x) + f'''(x)g(x) + 2\left(f'(x)g''(x) + f''(x)g'(x)\right)$$
$$+ f(x)g'''(x) + f'(x)g''(x)$$

{since the derivative of $[c \cdot f(x)]$ is $c \cdot f'(x)$}

$$\boxed{F'''(x) = f'''(x)g(x) + 3f''(x)g'(x) + 3f'(x)g''(x) + f(x)g'''(x)}$$

We know $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. Using powers to denote the derivative and $a = f(x)$, $b = g(x)$ as before, $F'''(x)$ matches the form of $(a+b)^3$.

47. $g(x) = [f(x)]^2 = f(x) \cdot f(x)$

According to the product rule,

$$g'(x) = f(x) \cdot f'(x) + f'(x) \cdot f(x)$$

$$\boxed{g'(x) = 2f(x)f'(x)}$$

This is the desired form ☺

0.7: 17. $\sin^3 x$

Let $f(g) = g^3$ and $g(x) = \sin x$

Then $f \circ g(x) = f(g(x)) = f(\sin x) = (\sin x)^3 = \sin^3 x$ ✓

26. $\ln 3x - 5$

Let $f(g) = g - 5$ and $g(x) = \ln 3x$

Then $f \circ g(x) = f(g(x)) = f(\ln 3x) = \ln 3x - 5$ ✓

2.7: 1. Fred's rate of motion: $F' = 10$ mph

Greg's rate of motion: $G' = 2F'$

Or, in other words, Greg can run twice as fast as Fred, so that (in distance) $G = 2F$ and $G' = 2F'$

In this context, the chain rule is obvious (!!).

31. $f(x) = \sin(\ln(\cos x^3))$

$$f'(x) = \cos(\ln(\cos x^3)) \cdot \left(\frac{1}{\cos x^3}\right) \cdot (-\sin x^3) \cdot (3x^2)$$

$$\boxed{f'(x) = -3x^2 \cdot \tan x^3 \cdot \cos(\ln(\cos x^3))}$$

38. $f(x) = \sqrt{\dfrac{x \sin x}{x^2+4}} = \left(\dfrac{x \sin x}{x^2+4}\right)^{1/2}$

$$f'(x) = \frac{1}{2}\left(\frac{x \sin x}{x^2+4}\right)^{-1/2} \left(\frac{\{x^2+4\}(x\cos x + \sin x) - x\sin x\,(2x)}{(x^2+4)^2}\right)$$

chain rule & quotient rule w/ product & power rules

$$= \frac{1}{2}\sqrt{\frac{x^2+4}{x \sin x}} \left\{\frac{x^3\cos x + x^2 \sin x + 4x\cos x + 4\sin x - 2x^2 \sin x}{(x^2+4)^2}\right\}$$

$$\boxed{f'(x) = \frac{1}{2} \cdot \frac{x^3 \cos x - x^2 \sin x + 4x\cos x + 4\sin x}{(x \sin x)(x^2+4)^{3/2}}}$$

2.8:

2. In this problem, the derivative is being taken with respect to x, i.e. y' means $\frac{d}{dx}(y)$. Instead of thinking of implicit differentiation think of the chain rule. Then, differentiating both sides:

$$\frac{d}{dx}(x^2y^2+3) = \frac{d}{dx}(x)$$

(and using product rule as well)

$$= x^2 \cdot \frac{d}{dx}(y^2) + \frac{d}{dx}(x^2)\, y^2 + \frac{d}{dx}3 = \frac{d}{dx}x$$

$$= x^2\left(2y\frac{dy}{dx}\right) + 2x\frac{dx}{dx}y^2 + 0 = \frac{dx}{dx}$$

But $\frac{dx}{dx} = 1$:

$$= 2x^2y\frac{dy}{dx} + 2xy^2 = 1$$

So it appears that we "tack on" a y' and take "regular" derivatives of x while in fact we're taking all "regular" derivatives with the chain rule. $\frac{dx}{dx}$ goes away since $\frac{dx}{dx}=1$ and $\frac{dy}{dx}$ "stays around" since we don't know what $\frac{dy}{dx}$ is.

12. $\sin xy = x^2 - 3$

$\frac{d}{dx}(\sin xy) = \frac{d}{dx}(x^2-3)$ Differentiate both sides w.r.t. x

$\cos xy\left(\left(\frac{d}{dx}x\right)\cdot y + x\cdot\frac{d}{dx}y\right) = 2x\frac{dx}{dx} - 0$ Apply chain and product rules.

$\cos xy\left(1\cdot y + x\frac{dy}{dx}\right) = 2x$ Simplify

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$y \cdot \cos xy + x \cdot \cos xy \frac{dy}{dx} = 2x$ Simplify

$$\boxed{y' = \frac{dy}{dx} = \frac{2x - y \cdot \cos xy}{x \cdot \cos xy}}$$ Solve for $\frac{dy}{dx} = y'$

20. $e^{x^2} y - 3y = x^2 + 1$

$\frac{d}{dx}\left(e^{x^2} y - 3y\right) = \frac{d}{dx}(x^2+1)$ Differentiate both sides of the equation wrt x.

$e^{x^2} \cdot \frac{d}{dx} y + \frac{d}{dx}(e^{x^2}) \cdot y - 3 \frac{d}{dx} y = \frac{d}{dx}(x^2) + \frac{d}{dx}(1)$ "distribute" the derivative

$e^{x^2} \frac{dy}{dx} + e^{x^2} \cdot 2x \cdot \frac{dx}{dx} \cdot y - 3 \frac{dy}{dx} = 2x \cdot \frac{dx}{dx} + 0$ simplify

$e^{x^2} \frac{dy}{dx} + 2xy e^{x^2} - 3 \frac{dy}{dx} = 2x$

$(e^{x^2} - 3) \frac{dy}{dx} = 2x(1 - ye^{x^2})$

$$\boxed{y' = \frac{dy}{dx} = \frac{2x(1 - ye^{x^2})}{e^{x^2} - 3}}$$