Homework 4 SOLUTIONS

Set 1: CiC: 13, 14, 17, 18

Set 2: SM: 3.1: 2, 6, 15, 17

Set 3: SM: 1.1: 5, 7, 41; 1.2: 11, 26

BONUS: SM 1.2: 57, 58

Set 1: CiC 3.3: 13, 14, 17, 18

13(a) $y = f(x)$ s.t. $f(5) = 1.2$, $f'(5) = .4$

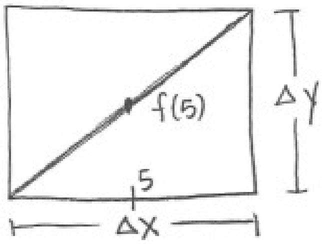
Microscope equation at x: $\Delta y = f'(x)\Delta x$

$\Rightarrow \boxed{\Delta y \approx .4\,\Delta x}$

or $f(x) - f(5) = .4(x-5)$

$\boxed{f(x) \approx f(5) + .4(x-5)}$

(b) graph:



You don't need to know what f(x) looks like since as long as the microscope is small enough, the function will look locally linear since its derivative $f'(5)$ exists.

(c) $x = 5.3 = 5 + .3$

(i) $\boxed{\Delta x \approx .3}$

(ii) $\boxed{\Delta y} = .4(.3) \boxed{\approx .12}$

(iii) $f(5.3) \approx f(5) + .4(.3) = 1.2 + .12$

$\boxed{f(5.3) \approx 1.32}$

(d) $f(5.23) \approx f(5) + .4(5.23 - 5) = 1.2 + .4(.23)$

$\boxed{f(5.23) \approx 1.292}$

$f(4.9) \approx f(5) + .4(4.9 - 5) = 1.2 + .4(-.1)$

$\boxed{f(4.9) \approx 1.16}$

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13(d) cont'd.

$$f(4.82) \approx f(5) + .4(4.82-5) = 1.2 + .4(-.18)$$

$$\boxed{f(4.82) \approx 1.128}$$

$$f(9) \approx f(5) + .4(9-5) = 1.2 + .4(4)$$

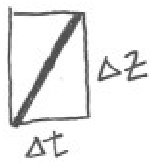
$$\boxed{f(9) \approx 2.8}$$

The approximation of $f(9)$ is not as reliable as the others since the distance between 9 and 5 is much larger than all the others.

14 (a) $z = g(t)$; $g(-4) = 7$; $g'(-4) = 3.5$

$$\boxed{\begin{array}{l} \Delta z \approx g'(-4)\,\Delta t \\ \quad \text{or} \quad g(t) = g(-4) + g'(-4)(t+4) \end{array}}$$

(b)



(c) $g(-4.2) \approx g(-4) + g'(-4)(-4.2+4)$

$$= 7 + 3.5(-.2)$$

$$\boxed{g(-4.2) \approx 6.3}$$

$$g(-3.75) \approx g(-4) + g'(-4)(-3.75+4)$$

$$= 7 + 3.5(.25)$$

$$\boxed{g(-3.75) \approx 7.875}$$

(d) $g(t) = 6 \approx g(-4) + g'(-4)(t+4)$

$$= 7 + 3.5t + 14 = 21 + 3.5t$$

$$-15 \approx 3.5t \Rightarrow \boxed{t \approx -4.29}$$

$$g(t) = 8.5 \approx g(-4) + g'(-4)(t+4)$$

$$= 7 + 3.5t + 14 = 21 + 3.5t$$

$$-12.5 \approx 3.5t \Rightarrow \boxed{t \approx -3.57}$$

17 (a) $D(5) = 30\text{ ft}; \; D'(5) = 5 \text{ ft/sec.}$

$D(5.1) \approx D(5) + D'(5)(5.1-5)$

$= 30\text{ ft} + \frac{5\text{ ft}}{\text{sec}}(.1\text{ sec}) = 30\text{ft} + .5\text{ ft}$

$\boxed{D(5.1) \approx 30.5\text{ ft}}$

$D(5.8) \approx D(5) + D'(5)(5.8-5)$

$= 30\text{ ft} + \frac{5\text{ft}}{\text{sec}}(.8\text{ sec}) = 30\text{ ft} + 4\text{ ft}$

$\boxed{D(5.8) \approx 34\text{ ft}}$

$D(4.7) \approx D(5) + D'(5)(4.7-5)$

$= 30\text{ ft} + \frac{5\text{ ft}}{\text{sec}}(-.3\text{ sec}) = 30\text{ ft} + -1.5\text{ ft}$

$\boxed{D(4.7) \approx 28.5\text{ ft.}}$

(b) $D(2.8) = 22\text{ ft}; \; D(3.1) = 26\text{ ft.}$

$D'(3) \approx \frac{D(2.8)-D(3.1)}{2.8s - 3.1s} = \frac{22\text{ ft} - 26\text{ ft}}{2.8\text{ s} - 3.1\text{ s}} = \frac{-4\text{ft}}{-.3\text{ s}} = 13\tfrac{1}{3}\text{ ft/s}$

$\boxed{D'(3) \approx 13\,1/3\text{ ft/sec}}$

18. (a) $f(3.2) \approx f(3) + f'(3)(3.2-3) = 2 + 4(.2) = 2.8$ $\boxed{f(3.2) \approx 2.8}$

(b) $g(6.6) \approx g(7) + g'(7)(6.6-7) = 6 + .3(-.4) = 5.88$ $\boxed{g(6.6) \approx 5.88}$

(c) $0 \approx h(1.6) + h'(1.6)(x-1.6) = 1 - 5(x-1.6)$

$= 1 - 5x + 8 = 9 - 5x$

$0 \approx 9 - 5x \Rightarrow 5x \approx 9 \Rightarrow \boxed{x \approx 9/5}$

(d) $.15 \approx F(2) + F'(2)(x-2) = 0 + .4(x-2) = .4x - .8$

$.95 \approx .4x \Rightarrow \boxed{x \approx 2.375}$

(e) $G(.4) \approx G(0) + G'(0)(.4-0)$

$1.6 \approx 2 + G'(0)(.4)$

$-.4 \approx G'(0)(.4)$

$\boxed{G'(0) \approx -1}$

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(f) $H(2.9) \approx H(3) + H'(3)(2.9-3)$

$-1 \approx -3 + H'(3)(-.1)$

$2 \approx H'(3)(-.1)$

$\boxed{-20 \approx H'(3)}$

Set 2: SM 3.1: 2, 6, 15, 17

2. The tangent line approximation at $x=8$ only "touches" the graph at $(8, f(8))$. Away from this point there is distance between the tangent line at $x=8$ and the graph. This distance increases as the distance from $x=8$ increases. This distance represents the error in the approximation produced by the graph.

6. $f(x) = (x+1)^{1/3}$, $x_0 = 0$

tangent line: $y = f(x_0) + f'(x_0)(x - x_0)$

$x_0 = 0$; $f(x_0) = f(0) = (0+1)^{1/3} = 1^{1/3} = 1$

$f'(x) = \frac{1}{3}(x+1)^{-2/3}$; $f'(x_0) = f'(0) = \frac{1}{3}(0+1)^{-2/3} = \frac{1}{3}(1)^{-2/3} = \frac{1}{3}$

$y = 1 + \frac{1}{3}(x-0) \Rightarrow \boxed{y = 1 + \frac{1}{3}x}$

15. $f(x) = \sqrt{4+x}$ $x_0 = 0$

Linear approx.: $y = f(x_0) + f'(x_0)(x - x_0)$

$x_0 = 0$ $f(x_0) = f(0) = \sqrt{4+0} = 2$

$f'(x) = \frac{1}{2}(4+x)^{-1/2} = \frac{1}{2\sqrt{4+x}}$ $f'(x_0) = f'(0) = \frac{1}{2\sqrt{4+0}} = \frac{1}{4}$

$y = 2 + \frac{1}{4}(x-0) = 2 + \frac{1}{4}x$

$\boxed{\text{So for "small" } x \text{ we see that } \sqrt{4+x} \approx 2 + \tfrac{1}{4}x}$

x	approximation	"true" value
.01	2.0025	2.002498
.1	2.025	2.024846
1	2.25	2.236

approx := $2 + \frac{1}{4}x$
true := $\sqrt{4+x}$

* For $x = .01$ and $x = .1$ the approximation is very good.

17. We want to estimate $\sin(1)$. We know $\sin(\pi/3)$ and $\frac{\pi}{3} \approx 1.05$ is really close to 1, so we use that:

$$\sin(1) \approx \sin(\pi/3) + [\sin(x)]'\Big|_{x=\pi/3}(1-\pi/3)$$
$$= \frac{\sqrt{3}}{2} + \cos(x)\Big|_{x=\frac{\pi}{3}}\left(\frac{3-\pi}{3}\right)$$
$$= \frac{\sqrt{3}}{2} + \frac{1}{2}\left(1-\frac{\pi}{3}\right) = \frac{1}{2}\left(\sqrt{3}+1-\frac{\pi}{3}\right) \approx .842$$

$$\boxed{\sin(1) \approx \frac{1}{2}(\sqrt{3}+1-\pi/3) \approx .842}$$

Set 3: SM 1.1: 5, 7, 41 ; 1.2: 11, 26

5. (a) $\boxed{\lim_{x\to 0^-} f(x) = -2}$ (b) $\boxed{\lim_{x\to 0^+} f(x) = 2}$

(c) $\boxed{\lim_{x\to 0} f(x) \text{ d.n.e.}}$ (d) $\boxed{\lim_{x\to 1^-} f(x) = 1}$

(e) $\boxed{\lim_{x\to -1} f(x) \approx \frac{1}{3}}$ (f) $\boxed{\lim_{x\to 2^-} f(x) = -1}$

(g) $\boxed{\lim_{x\to 2^+} f(x) \approx 3}$ (h) $\boxed{\lim_{x\to 2} f(x) \text{ d.n.e.}}$

(i) $\boxed{\lim_{x\to -2} f(x) \approx 1.5}$ (j) $\boxed{\lim_{x\to 3} f(x) = 2.5}$

7. $f(x) = \dfrac{x-1}{\sqrt{x}-1}$

$$f(1.5) = \frac{1.5-1}{\sqrt{1.5}-1} \approx 2.2247$$
$$f(1.1) = \frac{1.1-1}{\sqrt{1.1}-1} \approx 2.0488$$
$$f(1.01) = \frac{1.01-1}{\sqrt{1.01}-1} \approx 2.005$$
$$f(1.001) = \frac{1.001-1}{\sqrt{1.001}-1} \approx 2.0005$$

conjecture:
$\lim_{x\to 1^+} f(x) = 2$

SM 1.1, 7 contd

$$f(0.5) = \frac{0.5-1}{\sqrt{0.5}-1} \approx 1.7071$$

$$f(0.9) = \frac{0.9-1}{\sqrt{0.9}-1} \approx 1.9487$$

conjecture: $\lim_{x\to 1^-} f(x) = 2$

$$f(0.99) = \frac{0.99-1}{\sqrt{0.99}-1} \approx 1.9950$$

$$f(0.999) = \frac{0.999-1}{\sqrt{0.999}-1} \approx 1.9995$$

Assuming both conjectures are correct, $\lim_{x\to 1} f(x)$ exists and equals 2.

41. If a function $f(x)$ approaches a limit L as $x \to a$, the values of $f(x)$ must get <u>arbitrarily</u> close to L. In the case of $f(x) = x^2$, $\lim_{x\to 0} x^2$ cannot be equal to $-.01$ because the closest $f(x)$ gets to $-.01$ is 0, this is not arbitrarily close (or as close as I want to get).

1.2: 11. $\lim_{x\to 3} \frac{x^2-x-6}{x-3} = \lim_{x\to 3} \frac{\cancel{(x-3)}(x+2)}{\cancel{(x-3)}}$

$= \lim_{x\to 3} x+2 = \boxed{5}$

26. $f(x) = \begin{cases} 2x\cos x & x<0 \\ x^3+2x & x\geq 0 \end{cases}$

$\lim_{x\to 0^-} f(x) = 0$ $\quad$ $\lim_{x\to 0^+} f(x) = 0$

$\boxed{\therefore \lim_{x\to 0} f(x) = 0}$

* you can "plug in" here since $f(x)$ is continuous in each part. You could also check values (as in 1.1 #7) or look at the graph.

BONUS PROBLEMS: SM 1.2: 57, 58

57. $\lim_{x \to a} f(x) = L$

According to theorem 2.3, the limit of a products is the product of the limits. Therefore

$$\lim_{x \to a} [f(x)]^3 = \lim_{x \to a} [f(x) \cdot f(x) \cdot f(x)]$$

$$= \lim_{x \to a} f(x) \cdot \lim_{x \to a} f(x) \cdot \lim_{x \to a} f(x)$$

$$= L \cdot L \cdot L = L^3$$

$$\therefore \lim_{x \to a} [f(x)]^3 = L^3.$$

Similarly,

$$\lim_{x \to a} [f(x)]^4 = \lim_{x \to a} [f(x)]^3 \cdot f(x)$$

$$= \lim_{x \to a} [f(x)]^3 \cdot \lim_{x \to a} f(x)$$

$$= L^3 \cdot L = L^4$$

$$\therefore \lim_{x \to a} [f(x)]^4 = L^4.$$

58. If a result is true for n_0 and true in general for the $(n+1)$ case when true for (n), we can get to any case by $[([(n_0+1)+1]+1)+1]\ldots$ So it is then true for any case.

We know, if $\lim_{x \to a} f(x) = L$ that $\lim_{x \to a} [f(x)]^2 = L^2$. This is our base case. Now we assume its True that $\lim_{x \to a} [f(x)]^n = L^n$.

What about the $(n+1)$ case?

$$\lim_{x \to a} [f(x)]^{n+1} = \lim_{x \to a} \{[f(x)]^n \cdot f(x)\} = \lim_{x \to a} [f(x)]^n \cdot \lim_{x \to a} f(x) = L^n \cdot L = L^{n+1}$$

So by math induction, its true in any case.