Math 110 | **Basic Calculus I** | Fall 1998

Class 12

Wednesday September 30

The Babylonian Algorithm, Limits and Rates of Change

Successive Approximations to Find Square Roots — The Babylonian Algorithm. We have already seen one application of successive approximations in the S-I-R model where we obtained a sequence of values using sucessively smaller Δt. Here is another. Pretend you don't have a square root key on your calculator. (The algorithm given here is quite ancient.) How can we approximate $\sqrt{2}$?

Suppose

$$\begin{aligned} a &= \sqrt{2}. \text{ Square both sides.} \\ a^2 &= 2 \text{ Divide both sides by } x. \\ a &= 2/a \end{aligned}$$

Only the true square root of 2 satisfies $\sqrt{2} = 2/\sqrt{2}$.

If x is an estimate which is less than the true value for $\sqrt{2}$ then $2/x$ is an estimate which is ________________ than the true value.

If x is an estimate which is greater than the true value for $\sqrt{2}$ then $2/x$ is an estimate which is ________________ than the true value.

Hence a better estimate will be __________________________.

Group Work:

Each team will estimate the square root of one of the first 8 prime numbers. Begin with $x = 1$ as an estimate for $\sqrt{p}$. Use successive approximations to find the value of $\sqrt{p}$ to **3 decimal places.** How many steps does it take?

# steps	**Guess**	**Approximation**
n	a_n	a_{n+1}

General Babylonian Algorithm.

STEP 1: Let a_0 be your initial estimate for $\sqrt{r}$.

STEP 2: Then the next estimate is the average of your most recent estimate and r divided by your most recent estimate.

$$a_{\text{new}} = \frac{a_{\text{old}} + \dfrac{r}{a_{\text{old}}}}{2}$$

STEP 3: Continue calculating terms in the sequence until you reach the level of accuracy desired.

NOTE: You may want to think about how you could use Microsoft Excel or your TI-83 to implement the Babylonian Algorithm. There is a TruBasic implementation in the Calculus directory called `BABYLON.TRU`.

Limits
The concept of a *limit* is a central theme of all Calculus courses. When using the Babylonian Algorithm to estimate $\sqrt{a}$, the sequence of successive approximations

$$\begin{aligned} &x_1 \\ x_2 &= \frac{x_1 + \frac{a}{x_1}}{2} \\ x_3 &= \frac{x_2 + \frac{a}{x_2}}{2} \\ &\vdots \\ x_n &= \frac{x_{n-1} + \frac{a}{x_{n-1}}}{2} \\ &\vdots \end{aligned}$$

stabilizes. We say that the *limit* of the sequence of successive approximations is $\sqrt{a}$, or, rather, that

$$\lim_{n\to\infty} x_n = \sqrt{a}.$$

Average and Instantaneous Velocity
Recall our discussion in *Class 11* about estimating velocity given a table (and a graph) of distance travelled versus time elapsed. Suppose the position of a car at time t seconds is given by (the function) $s(t)$ feet.

Then the **average velocity** of the car between $t = a$ and $t = b$ is given by,

$$v_{\text{ave}} = \frac{s(b) - s(a)}{b - a}.$$

The **instantaneous velocity** at $t = a$ is defined by,

$$v = \lim_{h\to 0} \frac{s(a+h) - s(a)}{h}.$$

Finding the rate of change of a linear function.
Previously, the rates of change of functions were the known quantity and we investigated an unknown function by applying Euler's method. (for example; we used S', I', and R' to investigate the functions S, I, and R in the SIR model.) Now we will begin with a known function values and investigate its rate of change.

The *average rate of change of a function $y = f(x)$ on an interval* $[a, b]$ is given by the change in the output divided by the change in the input:

$$\frac{\Delta y}{\Delta x} = \frac{\text{change in output}}{\text{change in input}} = \frac{f(b) - f(a)}{b - a}.$$

Example:
What is the average rate of change of the function $f(x) = 3x + 2$ on the interval $[4, 10]$?

What is the instantaneous rate of change of the function $f(x) = 3x + 2$ on the

Rates and slopes are really the same thing for linear functions.
For a **linear function**, the *instantaneous rate of change* (slope of the line) equals the *average rate of change*.

ANNOUNCEMENTS

Your Lab #1 Reports are due **tomorrow** in your lab section
Assignment: In *CiC*, Read sections 3.1 and 3.2 and **DO problems 7, 8 and 9 on pp. 104–105**. Due on Friday October 2.