Basic Calculus I

Class 8

Friday September 18

Analyzing Solutions Computed Using Euler's Method

Introduction to Successive Approximations

Consider the following rate equation and initial condition (a.k.a. an *initial value problem*)

$$y' = \frac{1}{2y}, \qquad y(1) = 1$$

which was Groupwork problem **(e)** from *Class 7*.

If we use Euler's Method *repeatedly*, with varying time steps Δt in order to estimate the value of $y(4)$ the table below summarizes the results:

Δt	$y(4)$
3	2.500000
1	2.106061
0.5	2.047627
0.25	2.022673
0.10	2.008821
0.01	2.000868
0.001	2.000087

Questions

Why is $\Delta t = 2$ missing from the table?

What do you suppose the **exact** value of $y(4)$ is?

Why are you confident that this is the actual value of $y(4)$?

A Note on Accuracy. For successive approximations where we get closer and closer estimates for some quantity, we take a commonsense approach to accuracy: we watch the digits stabilize. There is a problem with this, however. Suppose our calculations gave us the sequence of estimates

$$0.9,\ 0.99,\ 0.9999,\ 0.99999,\ 0.999999,\ 0.9999999, \text{ and so on.}$$

What value does it look like we are ultimately approaching?

In this case, not even the one's digit is "correct" even though the difference between the estimates is actually very small. To avoid this difficulty, we say that **an estimate a for some quantity r is accurate to p decimal places** if error (the absolute value of the difference between a and r) is less than $1/2 \times 10^{-p}$. That is if,

$$|r - a| < \frac{1}{2} \times 10^{-p}.$$

Accuracy to p decimal places	$\leftrightarrow$	Error less than
1		0.05
2		0.005
3		0.0005
$\vdots$		$\vdots$
p		$0.\underbrace{000\ldots0}_{p}5$

Piecewise-Linear Functions
Let's think some more about the same problem of trying to estimate the solution to

$$y' = \frac{1}{2y}, \qquad y(1) = 1$$

which we call $y(t)$, at $t = 4$
Let's confirm that using $\Delta t = 3$ produces the estimate $y(4) \approx 2.5$

Use $\Delta t = 2$

t	y	y'	Δy

Use $\Delta t = 3$

t	y	y'	Δy

Sketch the data from both tables on the graph below.

We can denote the approximation of $y(t)$ generated by Euler's Method using $\Delta t = 3$ by the symbol $\tilde{y}_{\Delta t=3}(t)$
Example
Let us write down $\tilde{y}_{\Delta t=3}(t)$ as a function of time on the domain of $[1, 4]$
What kind of function is it?
$\tilde{y}_{\Delta t=3}(t) =$

Let us also write down $\tilde{y}_{\Delta t=2}(t)$ as a function of time on the domain $[1, 5]$.
This kind of function is called a **piecewise linear function.**

$\tilde{y}_{\Delta t=2}(t) =$

Now we can use $\tilde{y}_{\Delta t=2}(t)$ to find an approximation to $y(4)$ using Euler's Method with $\Delta t = 2$.
$\tilde{y}_{\Delta t=2}(4) =$

ANNOUNCEMENTS

REMINDER Exam #1 is scheduled for **Thursday September 24** in your lab section
DO Quiz #3 and hand it in on **Mon Sep 21**.
Class is cancelled on **Fri Sep 25** for exam recovery!