Basic Calculus I

Class 32

Wednesday November 28

Visualizing Solutions To Rate Equations: Slope Fields

Solutions of Differential Equations and Initial Value Problems

The solution to a (differential) rate equation is a function, which when plugged into both sides of the equation, produces an equality.

In addition to a rate (differential) equation, one often has an **initial condition**. The combined problem (DE+IC) is then known as an Initial Value Problem (IVP). A solution to the IVP must satisfy (produce an equality when plugged into) **both** the Differential Equation AND the Initial Condition. Solutions to IVP (when they exist) are **unique**.

Exercise

Show that $M(t) = 2e^{3t}$ is a solution to the initial value problem $M' = 3M, \quad M(0) = 2$.

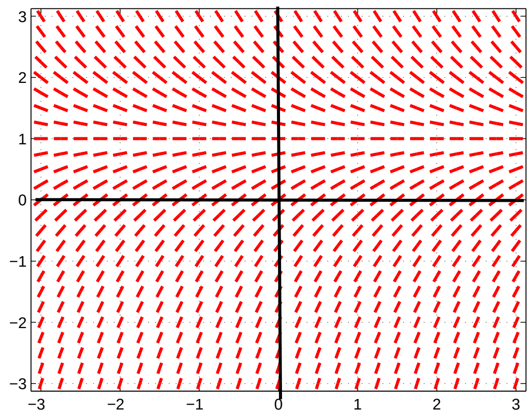
EXAMPLE

It turns out that the exact solution to the Newton's Law of Cooling initial value problem of $C' = k(C - A)$ with $C(0) = C_0$ can be calculated (through the magic of Calculus 2) and is $C(t) = A + (C_0 - A)e^{kt}$. Let's confirm that this function is indeed the solution of this IVP.

Slope Fields

A differential equation $\frac{dy}{dx} = f(x,y)$ or $y' = f(x,y)$ can be thought of as a function indicating the value of the slope to the unknown function $y(x)$ at every point (x,y). The graphical depiction of such a function is known as a **slope field**.

Let's consider the slope field $y' = 1 - y$ from *Anton, Bivens & Davis*, Homework 9.2, Number 3.



At every point in the xy-plane there is a little line representing a potential tangent or slope to the unknown function $y(x)$. The solution to the differential equation will start at the point in the plane reprepsented by the initial condition, i.e. $x = 0, y = -1$ and move tangentially to each of the little lines. Another way to think of the slope field is to think of it as a picture of running water with flecks of dirt in it and the solution curve will follow the path of the flecks of dirt.

There are numerous interesting things we can notice about this slope field:
1) All the little lines are angled the same way as we move from left to right on a horizontal path, in other words y' does not depend on x.
2) When $y = 1$ all the little lines are horizontal. What does that mean?
3) When $y > 1$ all solution curves will be DECREASING and CONCAVE UP.
4) When $y < 1$ all solution curves will be INCREASING and CONCAVE DOWN.
We could have predicted all of these features of the slope field from looking at the differential equation $y' = 1 - y$
1)
2)
3)
4)

$\boxed{\text{GroupWork}}$

Anton, Bivens & Davis, **Page 601, Number 9**. Match the differential equations with their corresponding slope fields.

(a) $y' = 1/x$ **(b)** $y' = 1/y$ **(b)**$y' = e^{-x^2}$

(d) $y' = y^2 - 1$ **(e)** $y' = \dfrac{x+y}{x-y}$ **(f)**$y' = \sin(x)\sin(y)$

FIGURE I

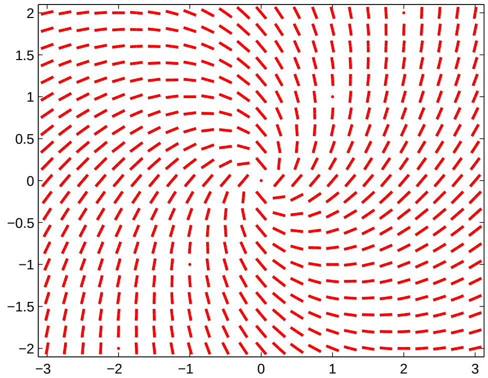


FIGURE II

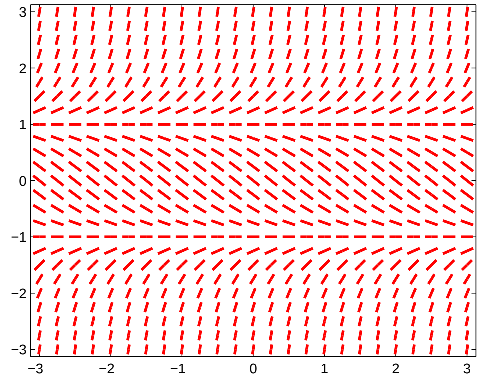


FIGURE III

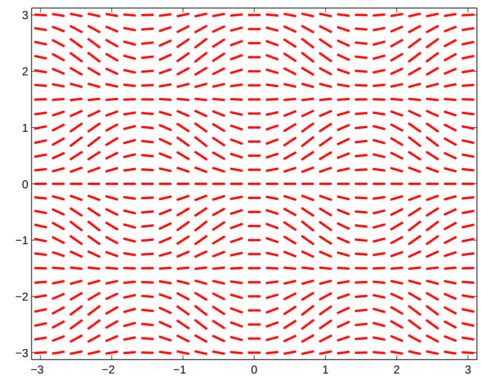


FIGURE IV

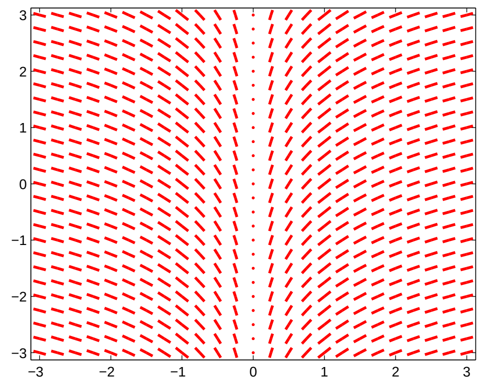


FIGURE V

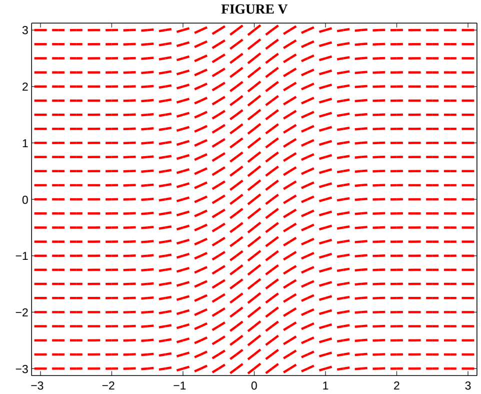
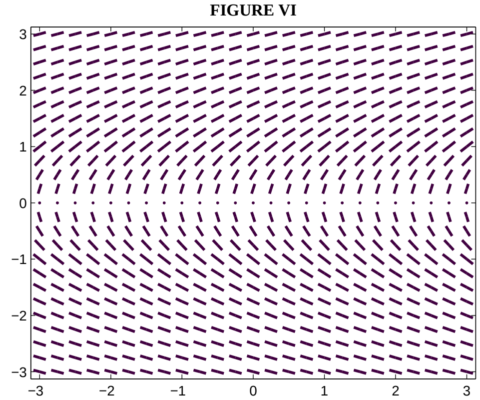
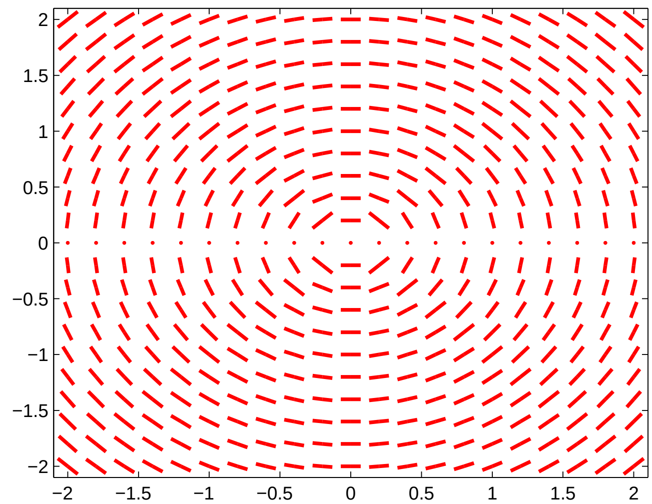


FIGURE VI



EXAMPLE

Consider the differential equation $y' = -x/y$ with initial condition $y(0) = 1$. Given that the exact solution is $y(x) = \sqrt{1-x^2}$,
(a) use the slope field to estimate $y(1/2)$ for the solution that satisfies the given initial condition.
(b) Compare your estimate with the exact value of $y(1)$ and
(c) Use Euler's Method with $\Delta x = .25$ to estimate $y(1/2)$.



To use Euler's Method generally the following table is helpful

x	y	y'	Δy