Math 110 | Fall 2007

Basic Calculus I

Class 31

Monday November 26

Euler's Method and Rate Equations

Modelling the growth of money.

Suppose Buckmire Bank pays a 5% return annually on funds deposited into a savings account. If your account contains \$300 at the beginning of the year, by how much has your account increased at the beginning of the next year?

Let $M(t)$ represent the amount of money in the account in dollars after t years. If M_0 represents the amount of money initially deposited into a savings account, by how much will M_0 increase after a year? What is the total sum of money in the account after a year?
$M(1) =$

After two years? $M(2) =$

After t years?

Let M' represent the rate at which the amount of money is increasing (in units of dollars per year). The equation:

$$M' =$$

is a *model* representing how your money grows in the bank.

For this model we are able to come up with a function $M(t)$ which tells us how much money we will have at some point t years in the future, if we know the rate $M'(t)$ at which money increases, and the amount of money M_0 we start with.
Write down this function $M(t) =$

Proportionality

In modelling, if we say that a quantity A varies proportionally to B then we denote this by $A \propto B$ and we can write the expression mathematically as $A = kB$ where k is called the constant of proportionality. In Chemistry, Boyle's Law states that the pressure of an ideal gas varies inversely proportionally to its volume when temperature is constant. This can be written $P \propto \dfrac{1}{V}$ or as an equation as $P = \dfrac{k}{V}$ or $PV = k$

Exercise

Can you think of other physical quantities that are proportional to each other? Write down the relationship and corresponding equation in the space below:

Newton's Law of Cooling
Newton's Law of Cooling states that the rate of cooling is proportional to the difference between the object's temperature and the ambient temperature. Let C denote the temperature of a hot liquid (say, coffee) in °F and let C' be the rate at which it is cooling (in °F per minute). Let the temperature of the room (ambient temperature) be denoted by A. Newton's Law of Cooling says:

$$C' \propto$$

If the coffee is at a temperature C which is larger than A, will the coffee's temperature go up or down as time goes on? And what does that tell you about C'?

If the coffee is at a temperature C less than the ambient temperature A, will the coffee get warmer or cooler with time? What does that tell you about C'?

Write an equation that relates C' and C. It will contain A, and a constant of proportionality k.

When the coffee is at 180 °F and the ambient temperature is 70 °F , the coffee is cooling at a rate of 9°F per minute. What is k?
Do you *expect* this value to be positive or negative? Now that you know k, wrie down the rate

equation model for how the coffee temperature changes with time

At what rate is the coffee's temperature changing after it has cooled down to 135 °F ?

If the temperature of the coffee is initially 180 °F , and it is cooling at 9°F per minute *estimate* its temperature C after 1 minute.

If the temperature of the coffee is initially 180 °F , *estimate* its temperature C after 5 minutes.

If the temperature of the coffee is initially 180 °F , *estimate* its temperature C after 10 and 20 minutes.

If the temperature of the coffee is initially C_0 °F , *estimate* its temperature C after t minutes.

How confident are you of your estimates of the coffee temperature? Do they "make sense"?

Our answers are <u>estimates</u> because the rate at which the coffee cools down **changes** as it is cooling down. In making our estimates, we are assuming that the rate of cooling down is constant over certain time intervals (1 minute, 5 minutes, or 10 minutes.) Unfortunately it is not!

(What time interval would give us the most accurate estimate?)

GROUPWORK

Which of the following gives a better estimate of the temperature after 10 minutes:

(a) We assume C' is constant for the entire ten minutes. Then $C(10)$ is obtained as we did above.

OR

(b) We assume that C' is constant for the first 5 minutes and calculate $C(5)$ as above. Then recalculate the rate of change, C', based on the new temperature and determine the change in temperature over the next 5 minutes based on the new rate of change. Add this change to the estimate for $C(5)$.

This idea of recalculating the rate of change frequently (every 5 minutes in this case) forms the basis of *Euler's Method* — a technique to approximate solutions to differential equations.

Euler's Method

Given an expression for how the derivative of an unknown function $y(x)$ changes, i.e. $y' = f(x, y)$, and an initial value $y(x_0) = y_0$ one can use Euler's Method to estimate $y(x)$ at any other point.

$y(x_{new}) = y(x_{old}) + \Delta y$ where $\Delta y \approx y'(x_{old})\Delta x$

This is often written as $y_{new} \approx y_{old} + y'_{old}\Delta x$

Exercise

1. **CONSTANT RATE OF CHANGE** Suppose a car is travelling at a CONSTANT speed of 80 miles/hr.
(a) How far will it travel in half an hour?

(b) How far will it travel in 20 seconds?

(c) How far will it travel in t hours?

2. **VARYING RATE OF CHANGE**. Suppose a car is travelling at 75 miles/hr. The driver applies the brakes. The speed of the car is given by $V(t) = 75 - 3t^2$, where time t is measured in seconds elapsed since the brakes were applied, and V is in miles/hr.

(a) What is the speed of the car when $t = 0$? When $t = 2.5$ seconds?

(b) How long does it take for the car to come to a stop?

(c) Use Euler's Method with one-second time intervals ($\Delta t = 1$) to estimate how far the car travels before coming to a stop.

(d) If we used half-second time intervals ($\Delta t = 0.5$) instead, would you expect to get an answer that is larger, smaller, or the same answer? Why? Was your original answer too big (an over-estimate) or too small (an under-estimate) of the exact distance travelled by the car? How do we know?

EXAMPLE

Using Euler's Method To Approximate Solutions To Differential Equations

(Differential Equation are also known as Rate of Change Equations or simply Rate Equations)

1. Suppose y changes with time t according to the equation $y' = 1 + 2y$.

(a) What is the rate of change of y when $y = 3$?
(b) Suppose when $t = 0$, $y = 3$. Use Euler's Method with $\Delta t = .5$ to estimate $y(1)$.
(c) Is your estimate of $y(1)$ an over-estimate or under-estimate?