Class 30

Monday November 19

Single Variable Optimization, Continued

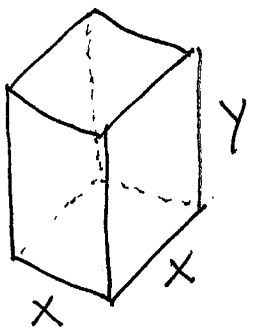
More Optimization Problems

In Groups of 3 or 4, work on the following problems. The ones in bold will be part of the graded homework.

Anton, Bivens & Davis §5.5: 16, **21**, 24, **25**, 26, **36**, 43, **44**, 55, **57**.

$\boxed{\text{GROUP 1}}$

Anton, Bivens & Davis, **Page 319, Question 21**. A closed rectangular container with a square base is to have a volume of 2000 cm^3. It costs twice as much per square centimeter for the top and bottom as it does for the sides. Find the dimension of the container of least cost.



$$x^2 y = 2000$$

$$C = 4xy(a) + 2x^2(2a)$$

cost per cm^2 = a, $y = \frac{2000}{x^2}$

$$C = 4ax\left(\frac{2000}{x^2}\right) + 4x^2 a$$

$$C = \frac{8000a}{x} + 4ax^2$$

$$\frac{dC}{dx} = -8000\frac{a}{x^2} + 8ax = 0$$

$$= 8a\left[x - \frac{1000}{x^2}\right] = 0$$

$$x = \frac{1000}{x^2} \Rightarrow x^3 = 1000$$

$$x = \sqrt[3]{1000}$$

$$x = 10$$

$$\frac{d^2C}{dx^2} = 8a + 16000\frac{a}{x^3}$$

$$(10)^2 y = 2000$$

$$y = \frac{2000}{100} = 20$$

At $x = 10$,

$$C''(10) = 8a + \frac{16000a}{1000}$$

$$= 24a > 0$$

$x = 10, y = 20$ is location of cost minimum

GROUP 2

Anton, Bivens & Davis, **Page 319, Question 25**. Find the dimensions of the right circular cylinder of greatest surface area that can be inscribed in a sphere of radius R.

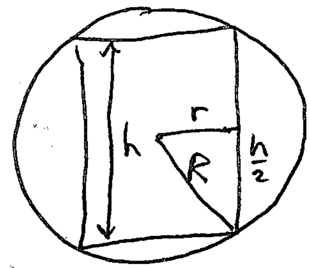
$$S = 2\pi r h + 2\pi r^2$$ (Surface area of cylinder)

$$V = \frac{4}{3}\pi R^3$$ Volume of sphere

$$r^2 + \left(\frac{h}{2}\right)^2 = R^2$$

$$h^2 = 4(R^2 - r^2)$$

$$h = 2\sqrt{R^2 - r^2}$$



$$S = 2\pi r 2\sqrt{R^2 - r^2} + 2\pi r^2$$

$$= 4\pi r\sqrt{R - r^2} + 2\pi r^2$$

$$\frac{dS}{dr} = 4\pi\sqrt{R^2 - r^2} + 4\pi r\frac{(-2r)}{2\sqrt{R^2 - r^2}} + 4\pi r$$

$$= 4\pi\sqrt{R^2 - r^2} - \frac{4\pi r^2}{\sqrt{R - r^2}} + 4\pi r$$

$$= 4\pi\left[\frac{R^2 - r^2 - r^2}{\sqrt{R^2 - r^2}} + r\right]$$

$$= 4\pi\left[\frac{R^2 - 2r^2}{\sqrt{R^2 - r^2}} + r\right] = 0$$

$$\frac{R^2 - 2r^2}{\sqrt{R^2 - r^2}} = -r$$

$$R^2 - 2r^2 = -r\sqrt{R^2 - r^2}$$

$$R^4 + 4r^4 - 4R^2r^2 = r^2(R^2 - r^2)$$

$$4r^4 + R^4 - 4R^2r^2 - r^2R^2 - r^4 = 0 \Rightarrow 5r^4 - 5r^2R^2 + R^4 = 0$$

Let $x = r^2$ $\longrightarrow$

$$5x^2 - 5R^2x + R^4 = 0$$

$$x = \frac{5R^2 \pm \sqrt{25R^4 - 4\cdot 5\cdot R^4}}{2\cdot 5}$$

$$= \frac{5R^2 \pm \sqrt{5R^4}}{10}$$

$$x = \frac{5R^2 \pm \sqrt{5}R^2}{10}$$

$$r^2 = x = \left(\frac{5 \pm \sqrt{5}}{10}\right)R^2 = \frac{5+\sqrt{5}}{10}R^2,\ \frac{5-\sqrt{5}}{10}R^2$$

$$\frac{d^2S}{dr^2} = 4\pi\left[1 + \frac{(-4r)\sqrt{R^2-r^2} - (R^2-2r^2)\left(\frac{1}{2}\frac{-2r}{\sqrt{R^2-r^2}}\right)}{R^2-r^2}\right]$$

$$= 4\pi\left[1 + \frac{(-4r)(R^2-r^2) + r(R^2-2r^2)}{(R^2-r^2)^{3/2}}\right]$$

$$= 4\pi$$

$$\frac{R^2-2r^2}{\sqrt{R^2-r^2}} = -r \Rightarrow \frac{R^2 - 2\left(\frac{5+\sqrt{5}}{10}\right)R^2}{\sqrt{R^2-\left(\frac{5+\sqrt{5}}{10}\right)R^2}} \overset{?}{=} -\sqrt{\left(\frac{5+\sqrt{5}}{10}\right)R^2}$$

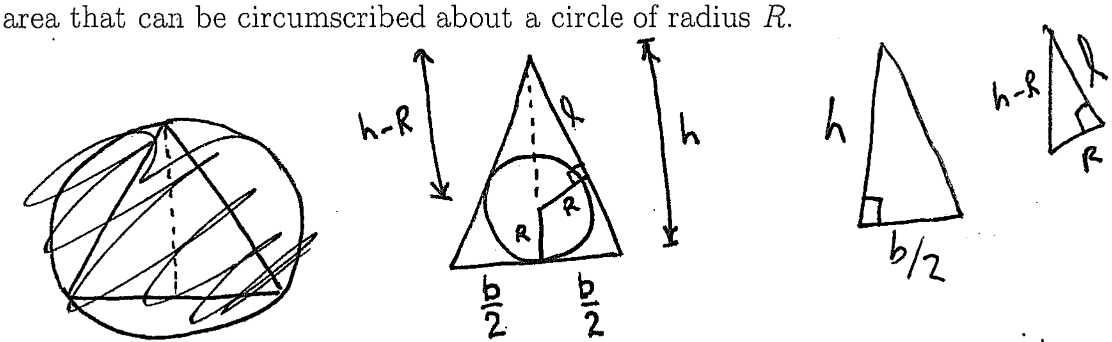
$$\frac{-\frac{2\sqrt{5}}{10}R^2}{\sqrt{\frac{5R^2+\sqrt{5}R^2}{10}}} \overset{?}{=} -\sqrt{\left(\frac{5+\sqrt{5}}{10}\right)R^2}$$

Only $r = \sqrt{\frac{5+\sqrt{5}}{10}}R$ satisfies the condition

$$h = 2\sqrt{R^2 - \left(\frac{5+\sqrt{5}}{10}\right)R^2} = \boxed{2R\sqrt{\frac{5-\sqrt{5}}{10}} = h} \quad \boxed{r = \sqrt{\frac{5+\sqrt{5}}{10}}R}$$

GROUP 3

Anton, Bivens & Davis, **Page 320, Question 36**. Find the dimensions of the isosceles triangle of least area that can be circumscribed about a circle of radius R.



$$R^2 + \ell^2 = (h-R)^2$$
$$\ell^2 = (h-R)^2 - R^2$$
$$= h^2 - 2hR + R^2 - R^2$$
$$\ell^2 = h^2 - 2hR$$
$$\ell = \sqrt{h^2 - 2hR}$$

By Similar Triangles

$$\frac{h}{b/2} = \frac{\ell}{R} = \frac{\sqrt{h^2-2hR}}{R}$$
$$\frac{2}{b} = \frac{\sqrt{h^2-2hR}}{Rh}$$
$$\frac{b}{2} = \frac{Rh}{\sqrt{h^2-2hR}}$$

$$\text{Area} = \frac{1}{2}bh = \left(\frac{Rh}{\sqrt{h^2-2hR}}\right)h$$

$$A(h) = \frac{Rh^2}{\sqrt{h^2-2hR}} = \frac{Rh^2}{\sqrt{h(h-2R)}}$$

Note $h - 2R \geq 0$ from the picture

Let's minimize A^2 instead of A

$$\cancel{A'(h) = 2hR} \quad A^2 = \frac{R^2h^4}{h(h-2R)} = \frac{R^2h^3}{h-2R}$$

$$\frac{d}{dh}(A^2) = \frac{(3h^2R^2)(h-2R) - R^2h^3(1)}{(h-2R)^2} = \frac{3h^3R^2 - 6h^2R^3 - R^2h^3}{(h-2R)^2}$$

$$= \frac{2h^3R^2 - 6h^2R^3}{(h-2R)^2} = \frac{2h^2R^2(h-3R)}{(h-2R)^2} = 0$$

C.P.'s at $h = 0, h = 2R, h = 3R$

$$\begin{array}{ccccccc} - & & - & & - & & + \\ & 0 & & 2R & & 3R & \end{array}$$

$h = 3R$ is location of a minimum

When $h = 3R$, $b = \dfrac{2R(3R)}{\sqrt{3R^2}} = 2\sqrt{3}R$

$$A = \frac{R(3R)^2}{\sqrt{3R \cdot R}} = \frac{9R^3}{R\sqrt{3}} = 3R^2\sqrt{3}$$

GROUP 4

Anton, Bivens & Davis, **Page 320, Question 44**. A firm determines that x units of is product can be sold daily at p dollars per unit, where $x = 1000 - p$. The cost of producing x units per day is $C(x) = 3000 + 20x$.

(a) Find the revenue function $R(x)$.

(b) Find the profit function $P(x)$.

(c) Assuming that the production capacity is at most 500 units per day, determine how many units the company must produce and sell each day to maximize the profit.

(d) Find the maximum profit.

(e) What price per unit must be charged to obtain the maximum profit.

(a) $R(x) = px$
$= (1000 - x)x$
$R(x) = 1000x - x^2$

(b) $P(x) = R(x) - C(x)$
$= 1000x - x^2 - [3000 + 20x]$
$P(x) = -x^2 + 980x - 3000$

(c) $0 \leq x \leq 500$

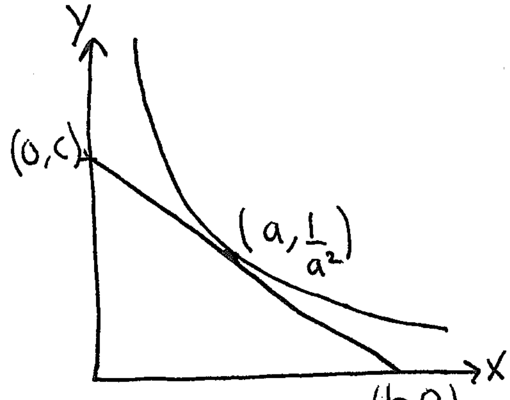
(d) $P'(x) = -2x + 980 = 0$
$980 = 2x$
$490 = x$

$P''(x) = -2$
$P''(490) = -2 < 0 \Rightarrow x = 490$ is location of a max

(e) $p = 1000 - x$
$= 1000 - 490$
$\boxed{p = 510}$

GROUP 5

Anton, Bivens & Davis, **Page 321, Question 55**. Find the coordinates of the point P on the curve $y = \frac{1}{x^2}$ $(x > 0)$ where the segment of the tangent line at P that is cut off by the coordinate axes has its shortest length.



$$L = \sqrt{b^2+c^2}$$

$$SL = b^2 + c^2$$

$$\boxed{c = \frac{3}{a^2}}$$

$$\frac{dy}{dx} = -\frac{2}{x^3}$$

$$x = a, \ y'(a) = -\frac{2}{a^3}$$

Eq^n of Tangent Line

$$y - \cancel{a}\frac{1}{a^2} = -\frac{2}{a^3}(x-a)$$

$$c = \frac{1}{a^2} - \frac{2}{a^3}(-a) = \frac{1}{a^2} + \frac{2}{a^2} = \frac{3}{a^2}$$

$$0 = \frac{1}{a^2} - \frac{2}{a^3}(b-a)$$

$$\left(\frac{a^3}{-2}\right)\left(-\frac{1}{a^2}\right) = b - a$$

$$\frac{a}{2} = b - a$$

$$\boxed{\frac{3a}{2} = b}$$

$$SL = \left(\frac{3a}{2}\right)^2 + \left(\frac{3}{a^2}\right)^2$$

$$= \frac{9a^2}{4} + \frac{9}{a^4}$$

$$\frac{d(SL)}{da} = \frac{18a}{4} - \frac{36}{a^5} = 0$$

$$\frac{9a}{2} = \frac{36}{a^5}$$

$$a^6 = \frac{72}{9} = 8$$

$$a^2 = 2$$

$$a = \sqrt{2}$$

$$\frac{d(SL)}{da^2} = \frac{18}{4} + \frac{180}{a^6}$$

$$SL''(\sqrt{2}) = \frac{9}{2} + \frac{180}{8}$$

$$= \frac{9}{2} + \frac{45}{2}$$

$$= \frac{54}{2} = 27 > 0$$

$(\sqrt{2}, \frac{1}{2})$ is the point minimizes the length SL^2.

GROUP 6

Anton, Bivens & Davis, **Page 321, Question 57**. Where on the curve $y = (1+x^2)^{-1}$ does the tangent line have the greatest slope?

$$\frac{dy}{dx} = -(1+x^2)^{-2} \cdot 2x = \frac{-2x}{(1+x^2)^2}$$

$$\frac{d^2y}{dx^2} = \frac{-2 \cdot (1+x^2)^2 - (-2x)2(1+x^2)2x}{(1+x^2)^4}$$

$$= \frac{-2 - 2x^2 + 8x^2}{(1+x^2)^3}$$

$$= \frac{6x^2 - 2}{(1+x^2)^3}$$

$$y'' = \frac{2(3x^2-1)}{(1+x^2)^3} = 0$$

$$3x^2 - 1 = 0 \Rightarrow x^2 = \frac{1}{3} \Rightarrow x = \pm\frac{1}{\sqrt{3}}$$

y'': $+$ on $\left(-\infty, -\frac{1}{\sqrt{3}}\right)$, $-$ on $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $+$ on $\left(\frac{1}{\sqrt{3}}, \infty\right)$

A max occurs at $x = -\frac{1}{\sqrt{3}}$, $y = \left(1 + \frac{1}{3}\right)^{-1} = \left(\frac{4}{3}\right)^{-1} = \frac{3}{4}$