Math 110 **Basic Calculus I** Fall 2007

Class 25

Wednesday November 7

L'Hôpital's Rule and Exotic Indeterminate Forms

THEOREM: L'Hôpital's Rule

If the limit on the left has an indeterminate form (i.e. $\frac{0}{0}$, $\frac{\pm\infty}{\pm\infty}$ or $\pm\infty \cdot 0$) then it is equal to the limit on the right (if this limit exists)

$$\lim_{x\to b}\frac{f(x)}{g(x)} = \frac{\lim_{x\to b} f'(x)}{\lim_{x\to b} g'(x)} \qquad \text{(L'Hôpital's Rule)}$$

It is very likely that the second limit may also have an indefinite form so L'Hôpital's Rule is often applied many many times repeatedly before a definitive value of the limt is obtained.

EXAMPLE

Take the following limits by first identifying which indeterminate form they take and then apply L'Hopital's Rule.

1. $\lim_{x\to 0}\frac{\sin(x)}{x}$

2. $\lim_{x\to 0}\frac{1-\cos(x)}{x}$

3. $\lim_{x\to 1}(x-1)^3\ln(x-1)$

4. $\lim_{x\to 0}\frac{\cos(x)-1+\frac{1}{2}x^2}{\sin(x)-x}$

By using this new rule we can find the limits of a whole bunch of new functions, and we have another way to find horizontal asymptotes of rational functions functions.

Exercise

1. $\lim_{x\to\infty}\frac{5+5x}{3x-2}$

2. $\lim_{x\to-\infty}\frac{5+5x}{3x-2}$

What is the limit $\lim_{x \to 0+} x^x = ?$ Let's first answer some easier questions.

(a) $0^1 =$ $\quad 0^{0.1} =$ $\quad 0^{0.01} =$ $\quad 0^{0.001} =$

(b) $1^0 =$ $\quad 0.1^0 =$ $\quad 0.01^0 =$ $\quad 0.001^0 =$

(c) $0^0 =$

So what can we conclude about $\lim_{x \to 0+} x^x$?

(d) $1^1 =$ $\quad 0.1^{0.1} =$ $\quad 0.01^{0.01} =$ $\quad 0.001^{0.001} =$

So what can we conclude about $\lim_{x \to 0+} x^x$?

Here's a solution for finding the answer without using a calculator and a table of values.:

(e) Warm-up: $e^{\ln(182)} =$ $\quad$ So $e^{\ln(x^x)} =$

Step 1. Write x^x as $e^{something}$:

Simplify: $\ln(x^x) =$ $\quad$ So $x^x = e^{\ln(x^x)} =$

So finding $\lim_{x \to 0+} x^x$ is the same as finding ____________________.

Step 2. $\lim_{x \to 0+} x \ln(x) =$

Step 3. $\lim_{x \to 0+} e^{x \ln(x)} =$

Exotic Indeterminate Forms: The following are also indeterminate forms ∞^0, 1^0, 1^∞, 0^∞ and 0^0.

These indeterminate forms require another approach before we can apply L'Hôpital's Rule. The following theorem is often helpful.

THEOREM

IF $f(x) > 0$, THEN $f(x)^{g(x)} = e^{\ln(f(x)^{g(x)})} = e^{g(x)\ln(f(x))}$.

In this case, $\lim_{x \to a} f(x)^{g(x)} = \lim_{x \to a} e^{g(x)\ln(f(x))} = e^{\lim_{x \to a} g(x)\ln(f(x))}$.

Exercise

Use the above to find $\lim_{x \to 0}(1+x)^{1/x}$.

Step 1. Write $(1+x)^{1/x}$ as $e^{something}$.

Step 2.

Step 3.