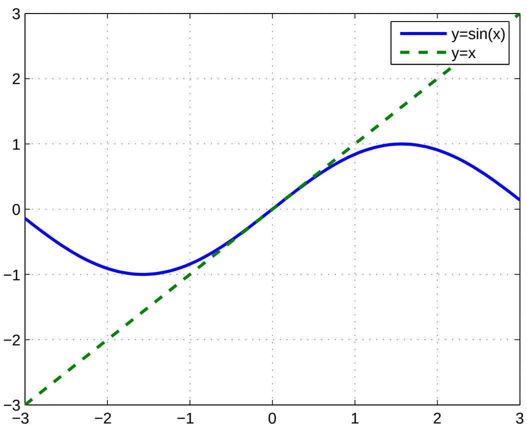
Math 110 — **Basic Calculus I** — Fall 2007

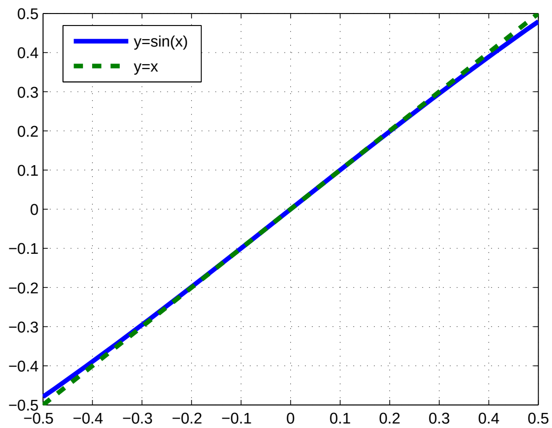
Class 20

Monday October 22

Approximations Using Derivatives

Recall the concept of **local linearity**, i.e. the notion that a function which is differentiable at a point also appears to be look like its tangent line the more you zoom in closer to the point. Consider the following diagrams of $f(x) = \sin(x)$ near $x = 0$





The tangent line to $f(x)$ at $x = x_0$ is $y = f(x_0) + f'(x_0)(x - x_0)$.

Q: What is the equation of the tangent line to $f(x) = \sin(x)$ at $x = 0$?
A: ____

Definition: Local Linear Approximation

The Local Linear Approximation to $f(x)$ near the point x_0 uses the idea that the function $f(x)$ is approximated by the tangent line to $y = f(x)$ at $x = x_0$, or

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

Sometimes this formula is re-written using $\Delta x = x - x_0$ so that $f(x_0 + \Delta x) \approx f'(x_0)\Delta x$.

The Microscope Approximation

Let $\Delta y = f(x) - f(x_0)$ and $\Delta x = x - x_0$, then the locally linear approximation can be re-written as $\Delta y \approx f'(x_0)\Delta x$

NOTE: the error $E(x)$ in the microscope approximation varries with x. It is given by the difference between $f(x)$ and its local linear approximation, $E(x) = f(x) - [f(x_0) + f'(x_0)(x - x_0)]$

EXAMPLE

Use the local linear approximation of $f(x) = \sqrt{x}$ at $x = 1$ to approximate $\sqrt{1.1}$.

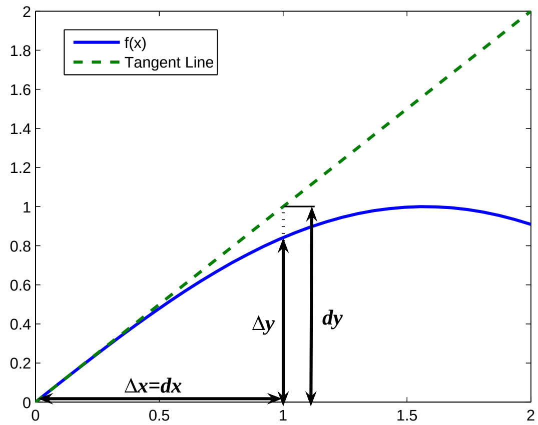
Exercise

Use a local linear approximation of $f(x) = \sqrt{x}$ to approximate $\sqrt{65}$

Differentials

Recall that the Leibniz Notation for the derivative function $f'(x)$ is $\frac{dy}{dx}$. If we consider that dy and dx are independent terms known as **differentials** then we can re-write $\frac{dy}{dx} = f'(x)$ as $dy = f'(x)dx$

Note that the differential dy and dx are very different from the increments Δx and Δy.



In the figure, Δy means the change in output of the *function* between the point the tangent line touches the function ,i.e. $(x_0, f((x_0))$, and any other point on the graph of the function.

dy means the change in output of the tangent line between $(x_0, f((x_0))$ and any other point. In this example, dx and Δx are set equal to each other, but they do not have to be equal in every situation.

Error Propagation

If the exact value of quantity is q but q has a measurement or calculation error Δq associated with q the relative error can be expressed as $\Delta q/q$. Since the actual EXACT value of q is usually not known the relative error in q is usually approximated by dq/q (and often expressed as a percentage) where dq is the differential quantity and q is the measured values.

EXAMPLE

Anton, Bivens & Davis **3.8.49** The electrical resistance R of a certain wire is given by $R = k/r^2$, where k is a constant and r is the radius of the wire. Assuming that the radius r has a possible error of $\pm 5\%$, use differentials to estimate the percentage error in R (assume k is exact).

Exercise

Anton, Bivens & Davis **3.8.52** The side of a square is measured with a possible percentage error of $\pm 1\%$. Use differentials to estimate the percentage error in the area.