Math 110 **BASIC CALCULUS I** Fall 2007

Class 16

Monday October 8

Understanding Differentiation

THEOREM: Differentiability Implies Continuity

IF a function $f(x)$ is differentiable at a point a THEN $f(x)$ is continuous at a.
This means that of $f'(a)$ exists, then $f(a)$ must exist and $\lim_{x\to a^-} f(x) = \lim_{x\to a^-} f(x) = f(a)$.

The **contrapositive** of a theorem is (always) also true: IF $f(x)$ is NOT continuous at a, THEN $f(x)$ is NOT differentiable at a.

1. **Local Linearity**
If the graph of a function at a point $x = a$ appears to look more and more like a line with finite slope when you zoom in on the point $(a, f(a))$ then the function is said to be **locally linear**. IF a function is locally linear at a point, THEN it is differentiable at that point. ALSO, IF a function is differentiable at a point, THEN it is locally linear at that point.

2. **Notations for The Derivative Function**
So far we have been using the notation $f'(x)$ to mean the function which outputs the instantaneous rate of change or derivative of f with respect to x.
If one thinks of the function $f(x)$ as an object then we can think of differentiation as an **operation** that is applied to the function $f(x)$ which produces a new function, the derivative of $f(x)$.

Often the notation $D_x[f(x)]$ or $\frac{d}{dx}[f(x)]$ to denote the derivative function. D_x and $\frac{d}{dx}$ are called the differentaion operator.
The relationship between the output (dependent) variable y and the input (independent) variable x is often represented as $y = f(x)$. In that case one can denote the derivative of y with respect to x as $\frac{dy}{dx}$ or $y'(x)$.

3. **Notations for The Value Of The Derivative At A Point**
Unfortunately, the notation for the value of the derivative of a function $f(x)$ with respect to x at a point a can get quote cumbersome. The most elegant notation is what we have been using, which is $f'(a)$. However, the following notations are all equivalent:

$$f'(a) = \frac{d}{dx}[f(x)]\bigg|_{x=a} = D_x[f(x)]|_{x=a} = \frac{dy}{dx}\bigg|_{x=a} = y'(a)$$

They all mean the value of the derivative at the point $x = a$ and are equal to the slope of the tangent line to the graph of the function $f(x)$ at $x = a$.

4. **Formula for Equation of a Tangent Line**
The formula for the equation of the tangent line to a function at $x = a$ is $y = f(a) + f'(a)(x - a)$.

Let's summarize the derivative functions we currently know (In each case write down an example of a function of the appropriate type and its corresponding derivative):

5. **Derivative Of A Constant Function**
The derivative of a constant function is zero; i.e. when $f(x) = c$, $f'(x) = 0$

6. **Derivative Of A Linear Function**
The derivative of a linear function is constant; it equals the slope of the line. When $f(x) = mx+b$, $f'(x) = m$

7. **Derivative Of A Power Function**
When $f(x) = x^n$ where n is an integer, $f'(x) = nx^{n-1}$. This is known as **The Power Rule**.

EXAMPLE

We shall find the derivative of $y = x^n$ where n is a positive integer, algebraically.

8. **Basic Derivative Rules**

THEOREM

(a) **The derivative of a constant multiple of a function is a constant multiple of the derivative function**
$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$
(b) **The derivative of the sum of two function is the sum of the derivative functions**
$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$
(c) **The derivative of the difference of two function is the difference of the derivative functions**
$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$

Exercise

Find $\frac{dy}{dx}$, given $y = x^7 - 6x^3 + 4x - 16^2 + 2x^{-5}$

GroupWork

At what points, if any, does the graph of $y = x^3 - 3x + 4$ have a horizontal tangent line?

9. **Higher Derivatives**
Since the derivative function f' is itself a function one can also find *its derivative*. This new function is called the second derivative of f and can be denoted f''. This process can be repeated as often as desired. The number of times a function has been differentiated is called the **order** of the derivative.

$$f^{(n)}(x) = \frac{d^n y}{dx^n} = \frac{d^n}{dx}[f(x)], \qquad y'' = \frac{d}{dx}[y'] \text{ and } y^{(3)} = (y'')'$$