Basic Calculus I

Class 8

Monday September 17

Introduction to Limits

Successive Approximations to Find Square Roots: The Babylonian Algorithm.
The concept of **successive approximation** is key to fully understanding the notion of limit intuitively. One fun application of successive approximations is provided here. Its called **The Babylonian Algorithm** since it was known by the people of Mesopotamia thousands of years before Calculus was invented!
Pretend you don't have a square root key on your calculator. How would you approximate $\sqrt{2}$? Suppose

$$\begin{aligned} a &= \sqrt{2}. \text{ Square both sides.} \\ a^2 &= 2 \text{ Divide both sides by } x. \\ a &= \frac{2}{a} \end{aligned}$$

Only the actual square root of 2 satisfies $\sqrt{2} = 2/\sqrt{2}$. (Of course this is true for any other number, there's nothing special about the number 2 here).

EXAMPLE

If x is an estimate which is **less than** (the true value of) $\sqrt{2}$ then $\frac{2}{x}$ is an estimate which is __________________ (the true value of) $\sqrt{2}$.

If x is an estimate which is **greater than** (the true value of) $\sqrt{2}$ then $\frac{2}{x}$ is an estimate which is __________________ (the true value of) $\sqrt{2}$.

Hence an estimate for the actual value of $\sqrt{2}$ which is better than either x or $\frac{2}{x}$ would be ____________________________________.

GroupWork

Each team will estimate the square root of one of the first 8 prime numbers (2, 3, 5, 7, 11, 13, etc). Begin with $x = 1$ as an estimate for $\sqrt{r}$. Use successive approximations to find the value of $\sqrt{r}$ to 6 decimal places. How many steps did it take?

# steps	Approximation	# steps	Approximation

General Babylonian Algorithm.

STEP 1: Let a_0 be your initial estimate for $\sqrt{r}$.

STEP 2: Then the next estimate is the average of your most recent estimate and r divided by your most recent estimate.

$$a_{\text{new}} = \frac{a_{\text{old}} + \frac{r}{a_{\text{old}}}}{2}$$

STEP 3: Continue calculating terms in the sequence until you reach the level of accuracy desired.

Consider the sequence of numbers you get when using the Babylonian algorithm to approximate $\sqrt{17}$.

Step	Approximation
1	1.00000000000000
2	9.00000000000000
3	5.44444444444444
4	4.28344671201814
5	4.12610662758133
6	4.12310562561781
7	4.12310562561766

The Babylonian Algorithm produces a sequence of numbers x_n such that

$$\lim_{n\to\infty} x_n = \sqrt{r}$$

We say that "the sequence x_n converges to its limit $\sqrt{r}$."

DEFINITION

An informal definition of **the limit of a function** $f(x)$ **at a point** a is if by taking (input) values of x "sufficiently close" to a (but NEVER ACTUALLY EQUAL TO IT) the (output) values of $f(x)$ can be made as close to the number L as desired. Symbolically, this is denoted

$$\lim_{x\to a} f(x) = L \quad \text{or} \quad f(x) \to L \text{ as } x \to a$$

Exercise Conjecture the value of $\lim_{x\to 1} \dfrac{x-1}{\sqrt{x}-1}$ considering the following data

x	$f(x) = \dfrac{x-1}{\sqrt{x}-1}$
0.9	1.9486832981
0.99	1.9949874371
0.999	1.9994998749
0.9999	1.9999499988
0.99999	1.9999950000
1	???
1.0001	2.0000499988
1.001	2.0004998751
1.01	2.0049875621
1.1	2.0488088482

DEFINITION

An informal definition of **the limit of a function** $f(x)$ **as** x **approaches** a **from the right** is if by taking (input) values of x "sufficiently close" to a (but ALWAYS GREATER THAN IT) the (output) values of $f(x)$ can be made as close to the number L as desired. Symbolically, this is denoted

$$\lim_{x\to a^+} f(x) = L \qquad \text{or} \qquad f(x) \to L \text{ as } x \to a^+$$

An informal definition of **the limit of a function** $f(x)$ **as** x **approaches** a **from the left** is if by taking (input) values of x "sufficiently close" to a (but ALWAYS LESS THAN IT) the (output) values of $f(x)$ can be made as close to the number L as desired. Symbolically, this is denoted

$$\lim_{x\to a^-} f(x) = L \qquad \text{or} \qquad f(x) \to L \text{ as } x \to a^-$$

THEOREM

The (two-sided) limit of a function $f(x)$ at $x = a$ equals L IF AND ONLY IF both one-sided limits exist **and** equal the same number L. Symbolically,

$$\lim_{x \to a} f(x) = L \iff \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$$

CLICKER QUESTION

TRUE or FALSE. $\lim_{x \to a} f(x) = L$ means that if x_1 is closer to a than x_2 is, then $f(x_1)$ will be closer to L than $f(x_2)$ is. You receive one point for your answer TRUE or FALSE and four points for your reasoning. Note: To prove a statement to be TRUE you must show it is always true (in every case), To prove statement to be false you have to provide a single "counter-example" where the statement fails to be true.

CLICKER QUESTION

Suppose you're trying to evaluate $\lim_{x \to 0} f(x)$. You plug in values of $x = 0.1, 0.01, 0.001, 0.0001, 0.0001, \ldots$ and get $f(x) = 0$ for all of these values. In fact, you are told that $f\left(\frac{1}{10^n}\right) = 0$ for *every* n = 1, 2, 3, ... **TRUE or FALSE**: Given the above information, we know that $\lim_{x \to 0} f(x) = 0$

CLICKER QUESTION

The statement "**Whether or not** $\lim_{x \to a} f(x)$ **exists depends on how** $f(a)$ **is defined**" is TRUE

(a) Sometimes

(b) Always

(c) Never

(d) Can Not Be Determined From The Information Given

CLICKER QUESTION

If a function $f(x)$ is not defined at $x = a$,

(a) $\lim_{x \to a} f(x)$ can not exist

(b) $\lim_{x \to a} f(x)$ could be zero

(c) $\lim_{x \to a} f(x)$ must approach ∞

(d) None of the above