(b) For a fixed value of n we have, for $k = 1, 2, \ldots, n$, $y_k = y_{k-1} + y_{k-1}\dfrac{1}{n} = \dfrac{n+1}{n}y_{k-1}$ In particular $y_n = \dfrac{n+1}{n}y_{n-1} = \left(\dfrac{n+1}{n}\right)^2 y_{n-2} = \ldots = \left(\dfrac{n+1}{n}\right)^n y_0 = \left(\dfrac{n+1}{n}\right)^n$. Consequently, $\lim\limits_{n\to+\infty} y_n = \lim\limits_{n\to+\infty}\left(\dfrac{n+1}{n}\right)^n = e$, which is the (correct) value $y = e^x\Big|_{x=1}$.

EXERCISE SET 9.3

1. (a) $\dfrac{dy}{dt} = ky^2,\ y(0) = y_0, k > 0$

   (b) $\dfrac{dy}{dt} = -ky^2,\ y(0) = y_0, k > 0$

3. (a) $\dfrac{ds}{dt} = \dfrac{1}{2}s$

   (b) $\dfrac{d^2s}{dt^2} = 2\dfrac{ds}{dt}$

4. (a) $\dfrac{dv}{dt} = -2v^2$

   (b) $\dfrac{d^2s}{dt^2} = -2\left(\dfrac{ds}{dt}\right)^2$

5. (a) $\dfrac{dy}{dt} = 0.02y,\ y_0 = 10{,}000$

   (b) $y = 10{,}000e^{2t/100}$

   (c) $T = \dfrac{1}{0.02}\ln 2 \approx 34.657$ h

   (d) $45{,}000 = 10{,}000e^{2t/100}$, $t = 50\ln\dfrac{45{,}000}{10{,}000} \approx 75.20$ h

6. $k = \dfrac{1}{T}\ln 2 = \dfrac{1}{20}\ln 2$

   (a) $\dfrac{dy}{dt} = ((\ln 2)/20)y,\ y(0) = 1$

   (b) $y(t) = e^{t(\ln 2)/20} = 2^{t/20}$

   (c) $y(120) = 2^6 = 64$

   (d) $1{,}000{,}000 = 2^{t/20}$, $t = 20\dfrac{\ln 10^6}{\ln 2} \approx 398.63$ min

7. (a) $\dfrac{dy}{dt} = -ky,\ y(0) = 5.0\times 10^7$; $3.83 = T = \dfrac{1}{k}\ln 2$, so $k = \dfrac{\ln 2}{3.83} \approx 0.1810$

   (b) $y = 5.0\times 10^7 e^{-0.181t}$

   (c) $y(30) = 5.0\times 10^7 e^{-0.1810(30)} \approx 219{,}000$

   (d) $y(t) = (0.1)y_0 = y_0e^{-kt}$, $-kt = \ln 0.1$, $t = -\dfrac{\ln 0.1}{0.1810} = 12.72$ days

8. (a) $k = \dfrac{1}{T}\ln 2 = \dfrac{1}{140}\ln 2 \approx 0.0050$, so $\dfrac{dy}{dt} = -0.0050y,\ y_0 = 10.$

   (b) $y = 10e^{-0.0050t}$

   (c) 10 weeks = 70 days so $y = 10e^{-0.35} \approx 7$ mg.

   (d) $0.3y_0 = y_0e^{-kt}$, $t = -\dfrac{\ln 0.3}{0.0050} \approx 240.8$ days

9. $100e^{0.02t} = 10{,}000$, $e^{0.02t} = 100$, $t = \dfrac{1}{0.02}\ln 100 \approx 230$ days

10. $y = 10{,}000e^{kt}$, but $y = 12{,}000$ when $t = 5$ so $10{,}000e^{5k} = 12{,}000$, $k = \dfrac{1}{5}\ln 1.2$. $y = 20{,}000$ when $2 = e^{kt}$, $t = \dfrac{\ln 2}{k} = 5\dfrac{\ln 2}{\ln 1.2} \approx 19$, in the year 2017.

11. $y(t) = y_0 e^{-kt} = 10.0e^{-kt}$, $3.5 = 10.0e^{-k(5)}$, $k = -\dfrac{1}{5}\ln\dfrac{3.5}{10.0} \approx 0.2100$, $T = \dfrac{1}{k}\ln 2 \approx 3.30$ days

12. $y = y_0 e^{-kt}$, $0.7y_0 = y_0 e^{-5k}$, $k = -\dfrac{1}{5}\ln 0.7 \approx 0.07$

(a) $T = \dfrac{\ln 2}{k} \approx 9.90$ yr

(b) $y(t) \approx y_0 e^{-0.07t}$, $\dfrac{y}{y_0} \approx e^{-0.07t}$, so $e^{-0.07t} \times 100$ percent will remain.

13. (a) $k = \dfrac{\ln 2}{6} \approx 0.1155; y \approx 3e^{0.1155t}$

(b) $y(t) = 4e^{0.02t}$

(c) $y = y_0 e^{kt}$, $1 = y_0 e^{k}$, $200 = y_0 e^{10k}$. Divide: $200 = e^{9k}$, $k = \dfrac{1}{9}\ln 200 \approx 0.5887$, $y \approx y_0 e^{0.5887t}$; also $y(1) = 1$, so $y_0 = e^{-0.5887} \approx 0.5550, y \approx 0.5550e^{0.5887t}$.

(d) $k = \dfrac{\ln 2}{T} \approx 0.1155$, $2 = y(1) \approx y_0 e^{0.1155}$, $y_0 \approx 2e^{-0.1155} \approx 1.7818$, $y \approx 1.7818e^{0.1155t}$

14. (a) $k = \dfrac{\ln 2}{T} \approx 0.1386$, $y \approx 10e^{-0.1386t}$

(b) $y = 10e^{-0.015t}$

(c) $100 = y_0 e^{-k}$, $1 = y_0 e^{-10k}$. Divide: $e^{9k} = 100$, $k = \dfrac{1}{9}\ln 100 \approx 0.5117$; $y_0 = e^{10k} \approx e^{5.117} \approx 166.83$, $y = 166.83e^{-0.5117t}$.

(d) $k = \dfrac{\ln 2}{T} \approx 0.1386$, $10 = y(1) \approx y_0 e^{-0.1386}$, $y_0 \approx 10e^{0.1386} \approx 11.4866$, $y \approx 11.4866e^{-0.1386t}$

16. (a) None; the half-life is independent of the initial amount.

(b) $kT = \ln 2$, so T is inversely proportional to k.

17. (a) $T = \dfrac{\ln 2}{k}$; and $\ln 2 \approx 0.6931$. If k is measured in percent, $k' = 100k$, then $T = \dfrac{\ln 2}{k} \approx \dfrac{69.31}{k'} \approx \dfrac{70}{k'}$.

(b) 70 yr

(c) 20 yr

(d) 7%

18. Let $y = y_0 e^{kt}$ with $y = y_1$ when $t = t_1$ and $y = 3y_1$ when $t = t_1 + T$; then $y_0 e^{kt_1} = y_1$ (i) and $y_0 e^{k(t_1+T)} = 3y_1$ (ii). Divide (ii) by (i) to get $e^{kT} = 3$, $T = \dfrac{1}{k}\ln 3$.

19. From (11), $y(t) = y_0 e^{-0.000121t}$. If $0.27 = \dfrac{y(t)}{y_0} = e^{-0.000121t}$ then $t = -\dfrac{\ln 0.27}{0.000121} \approx 10{,}820$ yr, and if $0.30 = \dfrac{y(t)}{y_0}$ then $t = -\dfrac{\ln 0.30}{0.000121} \approx 9950$, or roughly between 9000 B.C. and 8000 B.C.

end of nt intervals is $P(1+r/n)^{nt}$. ... and continuing in this fashion the value at the ... intervals it is

(b) Let $x = r/n$, then $n = r/x$ and
$\lim_{n\to+\infty} P(1+r/n)^{nt} = \lim_{x\to 0+} P(1+x)^{rt/x} = \lim_{x\to 0+} P[(1+x)^{1/x}]^{rt} = Pe^{rt}$.

(c) The rate of increase is $dA/dt = rPe^{rt} = rA$.

27. (a) $A = 1000e^{(0.08)(5)} = 1000e^{0.4} \approx \$1,491.82$

(b) $Pe^{(0.08)(10)} = 10,000$, $Pe^{0.8} = 10,000$, $P = 10,000e^{-0.8} \approx \$4,493.29$

(c) From (11), with $k = r = 0.08$, $T = (\ln 2)/0.08 \approx 8.7$ years.

28. Let r be the annual interest rate when compounded continuously and r_1 the effective annual interest rate. Then an amount P invested at the beginning of the year is worth $Pe^r = P(1+r_1)$ at the end of the year, and $r_1 = e^r - 1$.

29. (a) $\dfrac{dT}{dt} = -k(T-21)$, $T(0) = 95$, $\dfrac{dT}{T-21} = -k\,dt$, $\ln(T-21) = -kt + C_1$,
$T = 21 + e^{C_1}e^{-kt} = 21 + Ce^{-kt}$, $95 = T(0) = 21 + C, C = 74$, $T = 21 + 74e^{-kt}$

(b) $85 = T(1) = 21 + 74e^{-k}$, $k = -\ln\dfrac{64}{74} = -\ln\dfrac{32}{37}$, $T = 21 + 74e^{t\ln(32/37)} = 21 + 74\left(\dfrac{32}{37}\right)^t$,
$T = 51$ when $\dfrac{30}{74} = \left(\dfrac{32}{37}\right)^t$, $t = \dfrac{\ln(30/74)}{\ln(32/37)} \approx 6.22$ min

30. $\dfrac{dT}{dt} = k(70-T), T(0) = 40$; $-\ln(70-T) = kt + C$, $70 - T = e^{-kt}e^{-C}$, $T = 40$ when $t = 0$, so
$30 = e^{-C}$, $T = 70 - 30e^{-kt}$; $52 = T(1) = 70 - 30e^{-k}$, $k = -\ln\dfrac{70-52}{30} = \ln\dfrac{5}{3} \approx 0.5$,
$T \approx 70 - 30e^{-0.5t}$

31. Let T denote the body temperature of McHam's body at time t, the number of hours elapsed after 10:06 P.M.; then $\dfrac{dT}{dt} = -k(T-72)$, $\dfrac{dT}{T-72} = -kdt$, $\ln(T-72) = -kt + C, T = 72 + e^Ce^{-kt}$,
$77.9 = 72 + e^C$, $e^C = 5.9$, $T = 72 + 5.9e^{-kt}$, $75.6 = 72 + 5.9e^{-k}$, $k = -\ln\dfrac{3.6}{5.9} \approx 0.4940$,
$T = 72 + 5.9e^{-0.4940t}$. McHam's body temperature was last 98.6° when $t = -\dfrac{\ln(26.6/5.9)}{0.4940} \approx -3.05$,
so around 3 hours and 3 minutes before 10:06; the death took place at approximately 7:03 P.M., while Moore was on stage.

32. If $T_0 < T_a$ then $\dfrac{dT}{dt} = k(T_a - T)$ where $k > 0$. If $T_0 > T_a$ then $\dfrac{dT}{dt} = -k(T - T_a)$ where $k > 0$; both cases yield $T(t) = T_a + (T_0 - T_a)e^{-kt}$ with $k > 0$.

33. (a) Both $y(t) = 0$ and $y(t) = L$ are solutions of the logistic equation $\dfrac{dy}{dt} = k\left(1 - \dfrac{y}{L}\right)y$ as both sides of the equation are then zero.

(b) If y is very small relative to L then $y/L \approx 0$, and the logistic equation becomes $\dfrac{dy}{dt} \approx ky$, which is a form of the equation for exponential growth.

(c) All the terms on the right-hand-side of the logistic equation are positive, except perhaps $1 - \dfrac{y}{L}$, which is positive if $y < L$ and negative if $y > L$.

(d) The rate of change of y is a function of only one variable, y itself. The right-hand-side of the differential equation is a quadratic equation in y, which can be thought of as a parabola in y which opens down and crosses the y-axis at $y = 0$ and $y = L$. The parabola thus takes its maximum midway between the two y-intercepts, namely at $y = L/2$.

34. (a) Given $\dfrac{dy}{dt} = k\left(1 - \dfrac{y}{L}\right)y$, separation of variables yields $\dfrac{1}{L}\left(\dfrac{1}{y} + \dfrac{1}{L-y}\right)dy = \dfrac{k}{L}dt$ so that
$\ln y - \ln(L-y) = kt + C$. The initial condition yields
$\ln y_0 - \ln(L - \ldots$