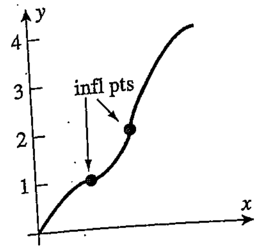
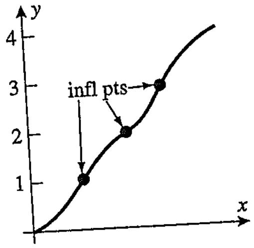
65.



66.



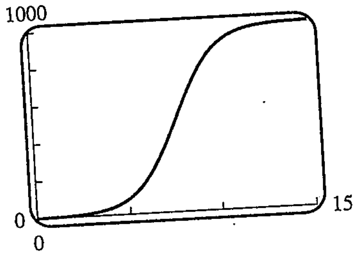
67. (a) $y'(t) = \dfrac{LAke^{-kt}}{(1+Ae^{-kt})^2}S$, so $y'(0) = \dfrac{LAk}{(1+A)^2}$

(b) The rate of growth increases to its maximum, which occurs when y is halfway between 0 and L, or when $t = \dfrac{1}{k}\ln A$; it then decreases back towards zero.

(c) From (2) one sees that $\dfrac{dy}{dt}$ is maximized when y lies half way between 0 and L, i.e. $y = L/2$. This follows since the right side of (2) is a parabola (with y as independent variable) with y-intercepts $y = 0, L$. The value $y = L/2$ corresponds to $t = \dfrac{1}{k}\ln A$, from (4).

68. Find t so that $N'(t)$ is maximum. The size of the population is increasing most rapidly when $t = 8.4$ years.

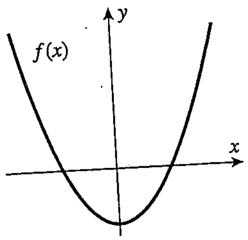
69. $t = 7.67$



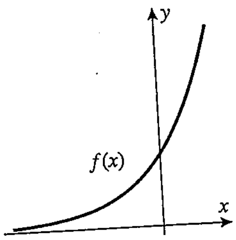
70. Since $0 < y < L$ the right-hand side of (3) of Example 9 can change sign only if the factor $L - 2y$ changes sign, which it does when $y = L/2$, at which point we have $\dfrac{L}{2} = \dfrac{L}{1+Ae^{-kt}}$, $1 = Ae^{-kt}$, $t = \dfrac{1}{k}\ln A$.

EXERCISE SET 5.2

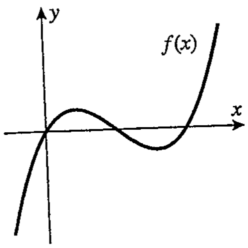
1. (a)



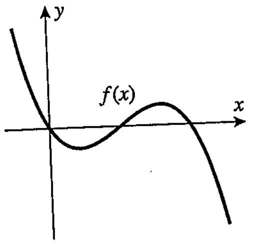
(b)



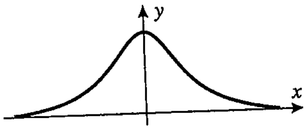
(c)



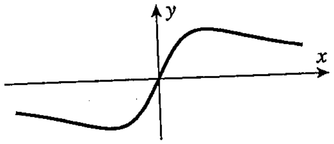
(d)



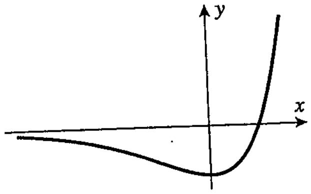
2. (a)



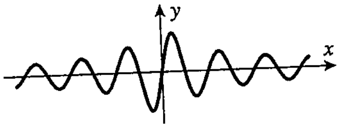
(b)



(c)



(d)



3. (a) $f'(x) = 6x - 6$ and $f''(x) = 6$, with $f'(1) = 0$. For the first derivative test, $f' < 0$ for $x < 1$ and $f' > 0$ for $x > 1$. For the second derivative test, $f''(1) > 0$.
   (b) $f'(x) = 3x^2 - 3$ and $f''(x) = 6x$. $f'(x) = 0$ at $x = \pm 1$. First derivative test: $f' > 0$ for $x < -1$ and $x > 1$, and $f' < 0$ for $-1 < x < 1$, so there is a relative maximum at $x = -1$, and a relative minimum at $x = 1$. Second derivative test: $f'' < 0$ at $x = -1$, a relative maximum; and $f'' > 0$ at $x = 1$, a relative minimum.

4. (a) $f'(x) = 2\sin x\cos x = \sin 2x$ (so $f'(0) = 0$) and $f''(x) = 2\cos 2x$. First derivative test: if x is near 0 then $f' < 0$ for $x < 0$ and $f' > 0$ for $x > 0$, so a relative minimum at $x = 0$. Second derivative test: $f''(0) = 2 > 0$, so relative minimum at $x = 0$.
   (b) $g'(x) = 2\tan x\sec^2 x$ (so $g'(0) = 0$) and $g''(x) = 2\sec^2 x(\sec^2 x + 2\tan^2 x)$. First derivative test: $g' < 0$ for $x < 0$ and $g' > 0$ for $x > 0$, so a relative minimum at $x = 0$. Second derivative test: $g''(0) = 2 > 0$, relative minimum at $x = 0$.
   (c) Both functions are squares, and so are positive for values of x near zero; both functions are zero at $x = 0$, so that must be a relative minimum.

5. (a) $f'(x) = 4(x-1)^3$, $g'(x) = 3x^2 - 6x + 3$ so $f'(1) = g'(1) = 0$.
   (b) $f''(x) = 12(x-1)^2$, $g''(x) = 6x - 6$, so $f''(1) = g''(1) = 0$, which yields no information.
   (c) $f' < 0$ for $x < 1$ and $f' > 0$ for $x > 1$, so there is a relative minimum at $x = 1$; $g'(x) = 3(x-1)^2 > 0$ on both sides of $x = 1$, so there is no relative extremum at $x = 1$.

6. (a) $f'(x) = -5x^4$, $g'(x) = 12x^3 - 24x^2$ so $f'(0) = g'(0) = 0$.
   (b) $f''(x) = -20x^3$, $g''(x) = 36x^2 - 48x$, so $f''(0) = g''(0) = 0$, which yields no information.
   (c) $f' < 0$ on both sides of $x = 0$, so there is no relative extremum there; $g'(x) = 12x^2(x-2) < 0$ on both sides of $x = 0$ (for x near 0), so again there is no relative extremum there.

7. $f'(x) = 16x^3 - 32x = 16x(x^2 - 2)$, so $x = 0, \pm\sqrt{2}$ are stationary points.

8. $f'(x) = 12x^3 + 12 = 12(x+1)(x^2 - x + 1)$, so $x = -1$ is the stationary point.

9. $f'(x) = \dfrac{-x^2 - 2x + 3}{(x^2+3)^2}$, so $x = -3, 1$ are the stationary points.

10. $f'(x) = -\dfrac{x(x^3-16)}{(x^3+8)^2}$, so stationary points at $x = 0, 2^{4/3}$.

11. $f'(x) = \dfrac{2x}{3(x^2-25)^{2/3}}$; so $x = 0$ is the stationary point.

12. $f'(x) = \dfrac{2x(4x-3)}{3(x-1)^{1/3}}$, so $x = 0, 3/4$ are the stationary points.

13. $f(x) = |\sin x| = \begin{cases} \sin x, & \sin x \ge 0 \\ -\sin x, & \sin x < 0 \end{cases}$ so $f'(x) = \begin{cases} \cos x, & \sin x > 0 \\ -\cos x, & \sin x < 0 \end{cases}$ and $f'(x)$ does not exist when $x = n\pi$, $n = 0, \pm 1, \pm 2, \cdots$ (the points where $\sin x = 0$) because $\lim\limits_{x\to n\pi^-} f'(x) \ne \lim\limits_{x\to n\pi^+} f'(x)$ (see Theorem preceding Exercise 61, Section 3.3). Now $f'(x) = 0$ when $\pm\cos x = 0$ provided $\sin x \ne 0$ so $x = \pi/2 + n\pi$, $n = 0, \pm 1, \pm 2, \cdots$ are stationary points.

14. When $x > 0$, $f'(x) = \cos x$, so $x = (n + \frac{1}{2})\pi, n = 0, 1, 2, \ldots$ are stationary points.
When $x < 0$, $f'(x) = -\cos x$, so $x = (n + \frac{1}{2})\pi, n = -1, -2, -3, \ldots$ are stationary points.
f is not differentiable at $x = 0$, so the latter is a critical point but not a stationary point.

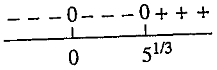
15. (a) none
(b) $x = 1$ because f' changes sign from + to − there
(c) none because $f'' = 0$ (never changes sign)

16. (a) $x = 1$ because $f'(x)$ changes sign from − to + there
(b) $x = 3$ because $f'(x)$ changes sign from + to − there
(c) $x = 2$ because $f''(x)$ changes sign there

17. (a) $x = 2$ because $f'(x)$ changes sign from − to + there.
(b) $x = 0$ because $f'(x)$ changes sign from + to − there.
(c) $x = 1, 3$ because $f''(x)$ changes sign at these points.

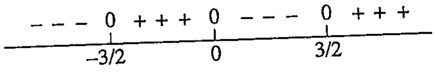
18. (a) $x = 1$ (b) $x = 5$ (c) $x = -1, 0, 3$

19. critical points $x = 0, 5^{1/3}$: f':



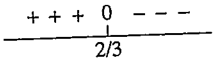
$x = 0$: neither
$x = 5^{1/3}$: relative minimum

20. critical points $x = -3/2, 0, 3/2$: f':



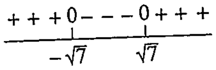
$x = -3/2$: relative minimum;
$x = 0$: relative maximum;
$x = 3/2$: relative minimum

21. critical points $x = 2/3$: f':



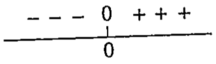
$x = 2/3$: relative maximum;

22. critical points $x = \pm\sqrt{7}$: f':



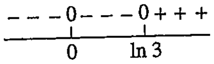
$x = -\sqrt{7}$: relative maximum;
$x = \sqrt{7}$: relative minimum

23. critical points $x = 0$: f':

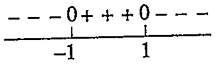


$x = 0$: relative minimum;

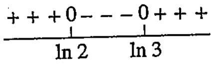
24. critical points $x = 0, \ln 3$: f':
$x = 0$: neither;
$x = \ln 3$: relative minimum



25. critical points $x = -1, 1$: f':
$x = -1$: relative minimum;
$x = 1$: relative maximum



26. critical points $x = \ln 2, \ln 3$: f':
$x = \ln 2$: relative maximum;
$x = \ln 3$: relative minimum



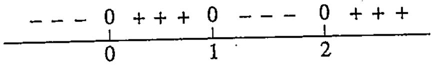
27. $f'(x) = 8 - 6x$: critical point $x = 4/3$
$f''(4/3) = -6$: f has a maximum of $19/3$ at $x = 4/3$

28. $f'(x) = 4x^3 - 36x^2$: critical points at $x = 0, 9$
$f''(0) = 0$: Theorem 5.2.5 with $m = 3$: f has an inflection point at $x = 0$
$f''(9) > 0$: f has a minimum of -2187 at $x = 9$

29. $f'(x) = 2\cos 2x$: critical points at $x = \pi/4, 3\pi/4$
$f''(\pi/4) = -4$: f has a maximum of 1 at $x = \pi/4$
$f''(3\pi/4) = 4$: f has a minimum of 1 at $x = 3\pi/4$

30. $f'(x) = (x - 2)e^x$: critical point at $x = 2$
$f''(2) = e^2$: f has a minimum of $-e^2$ at $x = 2$

31. $f'(x) = 4x^3 - 12x^2 + 8x$: critical points at $x = 0, 1, 2$
relative minimum of 0 at $x = 0$
relative maximum of 1 at $x = 1$
relative minimum of 0 at $x = 2$



32. $f'(x) = 4x^3 - 36x^2 + 96x - 64$; critical points at $x = 1, 4$
$f''(1) = 36$: f has a relative minimum of -27 at $x = 1$
$f''(4) = 0$: Theorem 5.2.5 with $m = 3$: f has an inflection point at $x = 4$

33. $f'(x) = 5x^4 + 8x^3 + 3x^2$: critical points at $x = -3/5, -1, 0$
$f''(-3/5) = 18/25$: f has a relative minimum of $-108/3125$ at $x = -3/5$
$f''(-1) = -2$: f has a relative maximum of 0 at $x = -1$
$f''(0) = 0$: Theorem 5.2.5 with $m = 3$: f has an inflection point at $x = 0$

34. $f'(x) = 5x^4 + 12x^3 + 9x^2 + 2x$: critical points at $x = -2/5, -1, 0$
$f''(-2/5) = -18/25$: f has a relative maximum of $108/3125$ at $x = -2/5$
$f''(-1) = 0$: Theorem 5.2.5 with $m = 3$: f has an inflection point at $x = -1$
$f''(0) = 2$: f has a relative maximum of 0 at $x = 0$

35. $f'(x) = \dfrac{2(x^{1/3} + 1)}{x^{1/3}}$: critical point at $x = -1, 0$

$f''(-1) = -\dfrac{2}{3}$: f has a relative maximum of 1 at $x = -1$

f' does not exist at $x = 0$. By inspection it is a relative minimum of 0.

36. $f'(x) = \dfrac{2x^{2/3} + 1}{x^{2/3}}$: no critical point except $x = 0$; since f is an odd function, $x = 0$ is an inflection point for f.