EXERCISE SET 4.3

1. (a) $f'(x) = 5x^4 + 3x^2 + 1 \geq 1$ so f is one-to-one on $-\infty < x < +\infty$.

(b) $f(1) = 3$ so $1 = f^{-1}(3)$; $\dfrac{d}{dx}f^{-1}(x) = \dfrac{1}{f'(f^{-1}(x))}$, $(f^{-1})'(3) = \dfrac{1}{f'(1)} = \dfrac{1}{9}$

2. (a) $f'(x) = 3x^2 + 2e^x$; for $-1 < x < 1$, $f'(x) \geq 2e^{-1} = 2/e$, and for $|x| > 1$, $f'(x) \geq 3x^2 \geq 3$, so f is increasing and one-to-one

(b) $f(0) = 2$ so $0 = f^{-1}(2)$; $\dfrac{d}{dx}f^{-1}(x) = \dfrac{1}{f'(f^{-1}(x))}$, $(f^{-1})'(2) = \dfrac{1}{f'(0)} = \dfrac{1}{2}$

3. $f^{-1}(x) = \frac{2}{x} - 3$, so directly $\frac{d}{dx}f^{-1}(x) = -\frac{2}{x^2}$. Using Formula (1),

$f'(x) = \frac{-2}{(x+3)^2}$, so $\frac{1}{f'(f^{-1}(x))} = -(1/2)(f^{-1}(x)+3)^2$,

$\frac{d}{dx}f^{-1}(x) = -(1/2)\left(\frac{2}{x}\right)^2 = -\frac{2}{x^2}$

4. $f^{-1}(x) = \frac{e^x - 1}{2}$, so directly, $\frac{d}{dx}f^{-1}(x) = \frac{e^x}{2}$. Next, $f'(x) = \frac{2}{2x+1}$, and using Formula (1),

$\frac{d}{dx}f^{-1}(x) = \frac{2f^{-1}(x)+1}{2} = \frac{e^x}{2}$

5. (a) $f'(x) = 2x+8$; $f' < 0$ on $(-\infty, -4)$ and $f' > 0$ on $(-4, +\infty)$; not enough information. By inspection, $f(1) = 10 = f(-9)$, so not one-to-one
   (b) $f'(x) = 10x^4 + 3x^2 + 3 \geq 3 > 0$; $f'(x)$ is positive for all x, so f is one-to-one
   (c) $f'(x) = 2 + \cos x \geq 1 > 0$ for all x, so f is one-to-one
   (d) $f'(x) = -(\ln 2)\left(\frac{1}{2}\right)^x < 0$ because $\ln 2 > 0$, so f is one-to-one for all x.

6. (a) $f'(x) = 3x^2 + 6x = x(3x+6)$ changes sign at $x = -2, 0$, so not enough information; by observation (of the graph, and using some guesswork), $f(-1+\sqrt{3}) = -6 = f(-1-\sqrt{3})$, so f is not one-to-one.
   (b) $f'(x) = 5x^4 + 24x^2 + 2 \geq 2 > 0$; f' is positive for all x, so f is one-to-one
   (c) $f'(x) = \frac{1}{(x+1)^2}$; f is one-to-one because:
   if $x_1 < x_2 < -1$ then $f' > 0$ on $[x_1, x_2]$, so $f(x_1) \neq f(x_2)$
   if $-1 < x_1 < x_2$ then $f' > 0$ on $[x_1, x_2]$, so $f(x_1) \neq f(x_2)$
   if $x_1 < -1 < x_2$ then $f(x_1) > 1 > f(x_2)$ since $f(x) > 1$ on $(-\infty, -1)$ and $f(x) < 1$ on $(-1, +\infty)$
   (d) Note that $f(x)$ is only defined for $x > 0$. $\frac{d}{dx}\log_b x = \frac{1}{x \ln b}$, which is always negative $(0 < b < 1)$, so f is one-to-one.

7. $y = f^{-1}(x)$, $x = f(y) = 5y^3 + y - 7$, $\frac{dx}{dy} = 15y^2 + 1$, $\frac{dy}{dx} = \frac{1}{15y^2+1}$;

check: $1 = 15y^2\frac{dy}{dx} + \frac{dy}{dx}$, $\frac{dy}{dx} = \frac{1}{15y^2+1}$

8. $y = f^{-1}(x)$, $x = f(y) = 1/y^2$, $\frac{dx}{dy} = -2y^{-3}$, $\frac{dy}{dx} = -y^3/2$;

check: $1 = -2y^{-3}\frac{dy}{dx}$, $\frac{dy}{dx} = -y^3/2$

9. $y = f^{-1}(x)$, $x = f(y) = 2y^5 + y^3 + 1$, $\frac{dx}{dy} = 10y^4 + 3y^2$, $\frac{dy}{dx} = \frac{1}{10y^4 + 3y^2}$;

check: $1 = 10y^4\frac{dy}{dx} + 3y^2\frac{dy}{dx}$, $\frac{dy}{dx} = \frac{1}{10y^4+3y^2}$

10. $y = f^{-1}(x)$, $x = f(y) = 5y - \sin 2y$, $\frac{dx}{dy} = 5 - 2\cos 2y$, $\frac{dy}{dx} = \frac{1}{5 - 2\cos 2y}$;

check: $1 = (5 - 2\cos 2y)\frac{dy}{dx}$, $\frac{dy}{dx} = \frac{1}{5 - 2\cos 2y}$

11. $7e^{7x}$

12. $-10xe^{-5x^2}$

13. $x^3e^x + 3x^2e^x = x^2e^x(x+3)$

14. $-\dfrac{1}{x^2}e^{1/x}$

15. $\dfrac{dy}{dx} = \dfrac{(e^x+e^{-x})(e^x+e^{-x}) - (e^x-e^{-x})(e^x-e^{-x})}{(e^x+e^{-x})^2}$

$= \dfrac{(e^{2x}+2+e^{-2x}) - (e^{2x}-2+e^{-2x})}{(e^x+e^{-x})^2} = 4/(e^x+e^{-x})^2$

16. $e^x\cos(e^x)$

17. $(x\sec^2 x + \tan x)e^{x\tan x}$

18. $\dfrac{dy}{dx} = \dfrac{(\ln x)e^x - e^x(1/x)}{(\ln x)^2} = \dfrac{e^x(x\ln x - 1)}{x(\ln x)^2}$

19. $(1-3e^{3x})e^{(x-e^{3x})}$

20. $\dfrac{15}{2}x^2(1+5x^3)^{-1/2}\exp(\sqrt{1+5x^3})$

21. $\dfrac{(x-1)e^{-x}}{1-xe^{-x}} = \dfrac{x-1}{e^x-x}$

22. $\dfrac{1}{\cos(e^x)}[-\sin(e^x)]e^x = -e^x\tan(e^x)$

23. $f'(x) = 2^x\ln 2$; $y = 2^x$, $\ln y = x\ln 2$, $\dfrac{1}{y}y' = \ln 2$, $y' = y\ln 2 = 2^x\ln 2$

24. $f'(x) = -3^{-x}\ln 3$; $y = 3^{-x}$, $\ln y = -x\ln 3$, $\dfrac{1}{y}y' = -\ln 3$, $y' = -y\ln 3 = -3^{-x}\ln 3$

25. $f'(x) = \pi^{\sin x}(\ln\pi)\cos x$;
$y = \pi^{\sin x}$, $\ln y = (\sin x)\ln\pi$, $\dfrac{1}{y}y' = (\ln\pi)\cos x$, $y' = \pi^{\sin x}(\ln\pi)\cos x$

26. $f'(x) = \pi^{x\tan x}(\ln\pi)(x\sec^2 x + \tan x)$;
$y = \pi^{x\tan x}$, $\ln y = (x\tan x)\ln\pi$, $\dfrac{1}{y}y' = (\ln\pi)(x\sec^2 x + \tan x)$
$y' = \pi^{x\tan x}(\ln\pi)(x\sec^2 x + \tan x)$

27. $\ln y = (\ln x)\ln(x^3-2x)$, $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{3x^2-2}{x^3-2x}\ln x + \dfrac{1}{x}\ln(x^3-2x)$,

$\dfrac{dy}{dx} = (x^3-2x)^{\ln x}\left[\dfrac{3x^2-2}{x^3-2x}\ln x + \dfrac{1}{x}\ln(x^3-2x)\right]$

28. $\ln y = (\sin x)\ln x$, $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{\sin x}{x} + (\cos x)\ln x$, $\dfrac{dy}{dx} = x^{\sin x}\left[\dfrac{\sin x}{x} + (\cos x)\ln x\right]$

29. $\ln y = (\tan x)\ln(\ln x)$, $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1}{x\ln x}\tan x + (\sec^2 x)\ln(\ln x)$,

$\dfrac{dy}{dx} = (\ln x)^{\tan x}\left[\dfrac{\tan x}{x\ln x} + (\sec^2 x)\ln(\ln x)\right]$

30. $\ln y = (\ln x)\ln(x^2+3)$, $\frac{1}{y}\frac{dy}{dx} = \frac{2x}{x^2+3}\ln x + \frac{1}{x}\ln(x^2+3)$,

$$\frac{dy}{dx} = (x^2+3)^{\ln x}\left[\frac{2x}{x^2+3}\ln x + \frac{1}{x}\ln(x^2+3)\right]$$

31. $f'(x) = ex^{e-1}$

32. (a) because x^x is not of the form a^x where a is constant

(b) $y = x^x$, $\ln y = x\ln x$, $\frac{1}{y}y' = 1 + \ln x$, $y' = x^x(1+\ln x)$

33. $\frac{3}{\sqrt{1-(3x)^2}} = \frac{3}{\sqrt{1-9x^2}}$

34. $-\frac{1/2}{\sqrt{1-\left(\frac{x+1}{2}\right)^2}} = -\frac{1}{\sqrt{4-(x+1)^2}}$

35. $\frac{1}{\sqrt{1-1/x^2}}(-1/x^2) = -\frac{1}{|x|\sqrt{x^2-1}}$

36. $\frac{\sin x}{\sqrt{1-\cos^2 x}} = \frac{\sin x}{|\sin x|} = \begin{cases} 1, & \sin x > 0 \\ -1, & \sin x < 0 \end{cases}$

37. $\frac{3x^2}{1+(x^3)^2} = \frac{3x^2}{1+x^6}$

38. $\frac{5x^4}{|x^5|\sqrt{(x^5)^2-1}} = \frac{5}{|x|\sqrt{x^{10}-1}}$

39. $y = 1/\tan x = \cot x$, $dy/dx = -\csc^2 x$

40. $y = (\tan^{-1} x)^{-1}$, $dy/dx = -(\tan^{-1} x)^{-2}\left(\frac{1}{1+x^2}\right)$

41. $\frac{e^x}{|x|\sqrt{x^2-1}} + e^x \sec^{-1} x$

42. $-\frac{1}{(\cos^{-1} x)\sqrt{1-x^2}}$

43. 0

44. $\frac{3x^2(\sin^{-1} x)^2}{\sqrt{1-x^2}} + 2x(\sin^{-1} x)^3$

45. 0

46. $-1/\sqrt{e^{2x}-1}$

47. $-\frac{1}{1+x}\left(\frac{1}{2}x^{-1/2}\right) = -\frac{1}{2(1+x)\sqrt{x}}$

48. $-\frac{1}{2\sqrt{\cot^{-1} x}(1+x^2)}$

49. (a) Let $x = f(y) = \cot y, 0 < y < \pi, -\infty < x < +\infty$. Then f is differentiable and one-to-one and $f'(f^{-1}(x)) = -\csc^2(\cot^{-1} x) = -x^2 - 1 \neq 0$, and

$$\frac{d}{dx}[\cot^{-1} x]\Big|_{x=0} = \lim_{x\to 0}\frac{1}{f'(f^{-1}(x))} = -\lim_{x\to 0}\frac{1}{x^2+1} = -1.$$

(b) If $x \neq 0$ then, from Exercise 50(a) of Section 1.5,

$\frac{d}{dx}\cot^{-1} x = \frac{d}{dx}\tan^{-1}\frac{1}{x} = -\frac{1}{x^2}\frac{1}{1+(1/x)^2} = -\frac{1}{x^2+1}$. For $x = 0$, Part (a) shows the same; thus for $-\infty < x < +\infty$, $\frac{d}{dx}[\cot^{-1} x] = -\frac{1}{x^2+1}$.

(c) For $-\infty < u < +\infty$, by the chain rule it follows that $\frac{d}{dx}[\cot^{-1} u] = -\frac{1}{u^2+1}\frac{du}{dx}$.

50. (a) By the chain rule, $\frac{d}{dx}[\csc^{-1} x] = \frac{d}{dx}\sin^{-1}\frac{1}{x} = -\frac{1}{x^2}\frac{1}{\sqrt{1-(1/x)^2}} = \frac{-1}{|x|\sqrt{x^2-1}}$

(b) By the chain rule, $\frac{d}{dx}[\csc^{-1} u] = \frac{du}{dx}\frac{d}{du}[\csc^{-1} u] = \frac{-1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$

51. $x^3 + x\tan^{-1} y = e^y$, $3x^2 + \frac{x}{1+y^2}y' + \tan^{-1} y = e^y y'$, $y' = \frac{(3x^2+\tan^{-1} y)(1+y^2)}{(1+y^2)e^y - x}$

52. $\sin^{-1}(xy) = \cos^{-1}(x-y)$, $\frac{1}{\sqrt{1-x^2y^2}}(xy'+y) = -\frac{1}{\sqrt{1-(x-y)^2}}(1-y')$,

$y' = \frac{y\sqrt{1-(x-y)^2} + \sqrt{1-x^2y^2}}{\sqrt{1-x^2y^2} - x\sqrt{1-(x-y)^2}}$

53. (a) $f(x) = x^3 - 3x^2 + 2x = x(x-1)(x-2)$ so $f(0) = f(1) = f(2) = 0$ thus f is not one-to-one.

(b) $f'(x) = 3x^2 - 6x + 2$, $f'(x) = 0$ when $x = \frac{6 \pm \sqrt{36-24}}{6} = 1 \pm \sqrt{3}/3$. $f'(x) > 0$ (f is increasing) if $x < 1 - \sqrt{3}/3$, $f'(x) < 0$ (f is decreasing) if $1 - \sqrt{3}/3 < x < 1 + \sqrt{3}/3$, so $f(x)$ takes on values less than $f(1-\sqrt{3}/3)$ on both sides of $1-\sqrt{3}/3$ thus $1-\sqrt{3}/3$ is the largest value of k.

54. (a) $f(x) = x^3(x-2)$ so $f(0) = f(2) = 0$ thus f is not one to one.

(b) $f'(x) = 4x^3 - 6x^2 = 4x^2(x-3/2)$, $f'(x) = 0$ when $x = 0$ or $3/2$; f is decreasing on $(-\infty, 3/2]$ and increasing on $[3/2, +\infty)$ so $3/2$ is the smallest value of k.

55. (a) $f'(x) = 4x^3 + 3x^2 = (4x+3)x^2 = 0$ only at $x = 0$. But on $[0,2]$, f' has no sign change, so f is one-to-one.

(b) $F'(x) = 2f'(2g(x))g'(x)$ so $F'(3) = 2f'(2g(3))g'(3)$. By inspection $f(1) = 3$, so $g(3) = f^{-1}(3) = 1$ and $g'(3) = (f^{-1})'(3) = 1/f'(f^{-1}(3)) = 1/f'(1) = 1/7$ because $f'(x) = 4x^3 + 3x^2$. Thus $F'(3) = 2f'(2)(1/7) = 2(44)(1/7) = 88/7$. $F(3) = f(2g(3)) = f(2\cdot 1) = f(2) = 24$, so the line tangent to $F(x)$ at $(3, 25)$ has the equation $y - 25 = (88/7)(x-3)$, $y = (88/7)x - 89/7$.

56. (a) $f'(x) = -e^{4-x^2}\left(2 + \frac{1}{x^2}\right) < 0$ for all $x > 0$, so f is one-to-one.

(b) By inspection, $f(2) = 1/2$, so $2 = f^{-1}(1/2) = g(1/2)$. By inspection,

$f'(2) = -\left(2 + \frac{1}{4}\right) = -\frac{9}{4}$, and

$$F'(1/2) = f'([g(x)]^2)\frac{d}{dx}[g(x)^2]\Big|_{x=1/2} = f'([g(x)]^2)2g(x)g'(x)\Big|_{x=1/2}$$

$$= f'(2^2)2\cdot 2\frac{1}{f'(g(x))}\Big|_{x=1/2} = 4\frac{f'(4)}{f'(2)} = 4\frac{e^{-12}(2+\frac{1}{16})}{(2+\frac{1}{4})} = \frac{33}{9e^{12}} = \frac{11}{3e^{12}}$$

57. (a) $f'(x) = ke^{kx}$, $f''(x) = k^2e^{kx}$, $f'''(x) = k^3e^{kx}, \ldots, f^{(n)}(x) = k^ne^{kx}$

(b) $g'(x) = -ke^{-kx}$, $g''(x) = k^2e^{-kx}$, $g'''(x) = -k^3e^{-kx}, \ldots, g^{(n)}(x) = (-1)^nk^ne^{-kx}$

58. $\frac{dy}{dt} = e^{-\lambda t}(\omega A\cos\omega t - \omega B\sin\omega t) + (-\lambda)e^{-\lambda t}(A\sin\omega t + B\cos\omega t)$

$= e^{-\lambda t}[(\omega A - \lambda B)\cos\omega t - (\omega B + \lambda A)\sin\omega t]$