CHAPTER 4

Derivatives of Logarithmic, Exponential, and Inverse Trigonometric Functions

EXERCISE SET 4.1

1. $y = (2x-5)^{1/3}$; $dy/dx = \frac{2}{3}(2x-5)^{-2/3}$

2. $dy/dx = \frac{1}{3}\left[2+\tan(x^2)\right]^{-2/3}\sec^2(x^2)(2x) = \frac{2}{3}x\sec^2(x^2)\left[2+\tan(x^2)\right]^{-2/3}$

3. $dy/dx = \frac{2}{3}\left(\frac{x+1}{x-2}\right)^{-1/3}\frac{x-2-(x+1)}{(x-2)^2} = -\frac{2}{(x+1)^{1/3}(x-2)^{5/3}}$

4. $dy/dx = \frac{1}{2}\left[\frac{x^2+1}{x^2-5}\right]^{-1/2}\frac{d}{dx}\left[\frac{x^2+1}{x^2-5}\right] = \frac{1}{2}\left[\frac{x^2+1}{x^2-5}\right]^{-1/2}\frac{-12x}{(x^2-5)^2} = -\frac{6x}{(x^2-5)^{3/2}\sqrt{x^2+1}}$

5. $dy/dx = x^3\left(-\frac{2}{3}\right)(5x^2+1)^{-5/3}(10x) + 3x^2(5x^2+1)^{-2/3} = \frac{1}{3}x^2(5x^2+1)^{-5/3}(25x^2+9)$

6. $dy/dx = -\frac{\sqrt[3]{2x-1}}{x^2} + \frac{1}{x}\frac{2}{3(2x-1)^{2/3}} = \frac{-4x+3}{3x^2(2x-1)^{2/3}}$

7. $dy/dx = \frac{5}{2}[\sin(3/x)]^{3/2}[\cos(3/x)](-3/x^2) = -\frac{15[\sin(3/x)]^{3/2}\cos(3/x)}{2x^2}$

8. $dy/dx = -\frac{1}{2}\left[\cos(x^3)\right]^{-3/2}\left[-\sin(x^3)\right](3x^2) = \frac{3}{2}x^2\sin(x^3)\left[\cos(x^3)\right]^{-3/2}$

9. (a) $1 + y + x\frac{dy}{dx} - 6x^2 = 0$, $\frac{dy}{dx} = \frac{6x^2-y-1}{x}$

(b) $y = \frac{2+2x^3-x}{x} = \frac{2}{x} + 2x^2 - 1$, $\frac{dy}{dx} = -\frac{2}{x^2} + 4x$

(c) From Part (a), $\frac{dy}{dx} = 6x - \frac{1}{x} - \frac{1}{x}y = 6x - \frac{1}{x} - \frac{1}{x}\left(\frac{2}{x} + 2x^2 - 1\right) = 4x - \frac{2}{x^2}$

10. (a) $\frac{1}{2}y^{-1/2}\frac{dy}{dx} - \cos x = 0$ or $\frac{dy}{dx} = 2\sqrt{y}\cos x$

(b) $y = (2+\sin x)^2 = 4 + 4\sin x + \sin^2 x$ so $\frac{dy}{dx} = 4\cos x + 2\sin x\cos x$

(c) from Part (a), $\frac{dy}{dx} = 2\sqrt{y}\cos x = 2\cos x(2+\sin x) = 4\cos x + 2\sin x\cos x$

11. $2x + 2y\frac{dy}{dx} = 0$ so $\frac{dy}{dx} = -\frac{x}{y}$

12. $3x^2 + 3y^2\frac{dy}{dx} = 3y^2 + 6xy\frac{dy}{dx}$, $\frac{dy}{dx} = \frac{3y^2-3x}{3y^2-6xy} = \frac{y^2-x^2}{y^2-2xy}$

13. $x^2\frac{dy}{dx} + 2xy + 3x(3y^2)\frac{dy}{dx} + 3y^3 - 1 = 0$

$(x^2+9xy^2)\frac{dy}{dx} = 1 - 2xy - 3y^3$ so $\frac{dy}{dx} = \frac{1-2xy-3y^3}{x^2+9xy^2}$

14. $x^3(2y)\frac{dy}{dx} + 3x^2y^2 - 5x^2\frac{dy}{dx} - 10xy + 1 = 0$

$(2x^3y - 5x^2)\frac{dy}{dx} = 10xy - 3x^2y^2 - 1$ so $\frac{dy}{dx} = \frac{10xy - 3x^2y^2 - 1}{2x^3y - 5x^2}$

15. $-\frac{1}{2x^{3/2}} - \frac{\frac{dy}{dx}}{2y^{3/2}} = 0,\ \frac{dy}{dx} = -\frac{y^{3/2}}{x^{3/2}}$

16. $2x = \frac{(x-y)(1+dy/dx) - (x+y)(1-dy/dx)}{(x-y)^2}$,

$2x(x-y)^2 = -2y + 2x\frac{dy}{dx}$ so $\frac{dy}{dx} = \frac{x(x-y)^2 + y}{x}$

17. $\cos(x^2y^2)\left[x^2(2y)\frac{dy}{dx} + 2xy^2\right] = 1,\ \frac{dy}{dx} = \frac{1 - 2xy^2\cos(x^2y^2)}{2x^2y\cos(x^2y^2)}$

18. $-\sin(xy^2)\left[y^2 + 2xy\frac{dy}{dx}\right] = \frac{dy}{dx},\ \frac{dy}{dx} = -\frac{y^2\sin(xy^2)}{2xy\sin(xy^2) + 1}$

19. $3\tan^2(xy^2+y)\sec^2(xy^2+y)\left(2xy\frac{dy}{dx} + y^2 + \frac{dy}{dx}\right) = 1$

so $\frac{dy}{dx} = \frac{1 - 3y^2\tan^2(xy^2+y)\sec^2(xy^2+y)}{3(2xy+1)\tan^2(xy^2+y)\sec^2(xy^2+y)}$

20. $\frac{(1+\sec y)[3xy^2(dy/dx) + y^3] - xy^3(\sec y\tan y)(dy/dx)}{(1+\sec y)^2} = 4y^3\frac{dy}{dx}$,

multiply through by $(1+\sec y)^2$ and solve for $\frac{dy}{dx}$ to get

$\frac{dy}{dx} = \frac{y(1+\sec y)}{4y(1+\sec y)^2 - 3x(1+\sec y) + xy\sec y\tan y}$

21. $4x - 6y\frac{dy}{dx} = 0,\ \frac{dy}{dx} = \frac{2x}{3y},\ 4 - 6\left(\frac{dy}{dx}\right)^2 - 6y\frac{d^2y}{dx^2} = 0,$

$\frac{d^2y}{dx^2} = -\frac{3\left(\frac{dy}{dx}\right)^2 - 2}{3y} = \frac{2(3y^2 - 2x^2)}{9y^3} = -\frac{8}{9y^3}$

22. $\frac{dy}{dx} = -\frac{x^2}{y^2},\ \frac{d^2y}{dx^2} = -\frac{y^2(2x) - x^2(2y\,dy/dx)}{y^4} = -\frac{2xy^2 - 2x^2y(-x^2/y^2)}{y^4} = -\frac{2x(y^3+x^3)}{y^5}$,

but $x^3 + y^3 = 1$ so $\frac{d^2y}{dx^2} = -\frac{2x}{y^5}$

23. $\frac{dy}{dx} = -\frac{y}{x},\ \frac{d^2y}{dx^2} = -\frac{x(dy/dx) - y(1)}{x^2} = -\frac{x(-y/x) - y}{x^2} = \frac{2y}{x^2}$

24. $y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0,\ \frac{dy}{dx} = -\frac{y}{x+2y},\ 2\frac{dy}{dx} + x\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} = 0,\ \frac{d^2y}{dx^2} = \frac{2y(x+y)}{(x+2y)^3}$

25. $\frac{dy}{dx} = (1+\cos y)^{-1},\ \frac{d^2y}{dx^2} = -(1+\cos y)^{-2}(-\sin y)\frac{dy}{dx} = \frac{\sin y}{(1+\cos y)^3}$

26. $\dfrac{dy}{dx} = \dfrac{\cos y}{1 + x\sin y}$,

$$\frac{d^2y}{dx^2} = \frac{(1 + x\sin y)(-\sin y)(dy/dx) - (\cos y)[(x\cos y)(dy/dx) + \sin y]}{(1 + x\sin y)^2}$$

$$= -\frac{2\sin y\cos y + (x\cos y)(2\sin^2 y + \cos^2 y)}{(1 + x\sin y)^3},$$

but $x\cos y = y$, $2\sin y\cos y = \sin 2y$, and $\sin^2 y + \cos^2 y = 1$ so

$$\frac{d^2y}{dx^2} = -\frac{\sin 2y + y(\sin^2 y + 1)}{(1 + x\sin y)^3}$$

27. By implicit differentiation, $2x + 2y(dy/dx) = 0$, $\dfrac{dy}{dx} = -\dfrac{x}{y}$; at $(1/2, \sqrt{3}/2)$, $\dfrac{dy}{dx} = -\sqrt{3}/3$; at $(1/2, -\sqrt{3}/2)$, $\dfrac{dy}{dx} = +\sqrt{3}/3$. Directly, at the upper point $y = \sqrt{1 - x^2}$, $\dfrac{dy}{dx} = \dfrac{-x}{\sqrt{1 - x^2}} = -\dfrac{1/2}{\sqrt{3/4}} = -1/\sqrt{3}$ and at the lower point $y = -\sqrt{1 - x^2}$, $\dfrac{dy}{dx} = \dfrac{x}{\sqrt{1 - x^2}} = +1/\sqrt{3}$.

28. If $y^2 - x + 1 = 0$, then $y = \sqrt{x - 1}$ goes through the point $(10, 3)$ so $dy/dx = 1/(2\sqrt{x - 1})$. By implicit differentiation $dy/dx = 1/(2y)$. In both cases, $dy/dx|_{(10,3)} = 1/6$. Similarly $y = -\sqrt{x - 1}$ goes through $(10, -3)$ so $dy/dx = -1/(2\sqrt{x - 1}) = -1/6$ which yields $dy/dx = 1/(2y) = -1/6$.

29. $4x^3 + 4y^3\dfrac{dy}{dx} = 0$, so $\dfrac{dy}{dx} = -\dfrac{x^3}{y^3} = -\dfrac{1}{15^{3/4}} \approx -0.1312$.

30. $3y^2\dfrac{dy}{dx} + x^2\dfrac{dy}{dx} + 2xy + 2x - 6y\dfrac{dy}{dx} = 0$, so $\dfrac{dy}{dx} = -2x\dfrac{y + 1}{3y^2 + x^2 - 6y} = 0$ at $x = 0$

31. $4(x^2 + y^2)\left(2x + 2y\dfrac{dy}{dx}\right) = 25\left(2x - 2y\dfrac{dy}{dx}\right)$,

$\dfrac{dy}{dx} = \dfrac{x[25 - 4(x^2 + y^2)]}{y[25 + 4(x^2 + y^2)]}$; at $(3, 1)$ $\dfrac{dy}{dx} = -9/13$

32. $\dfrac{2}{3}\left(x^{-1/3} + y^{-1/3}\dfrac{dy}{dx}\right) = 0$, $\dfrac{dy}{dx} = -\dfrac{y^{1/3}}{x^{1/3}} = \sqrt{3}$ at $(-1, 3\sqrt{3})$

33. $4a^3\dfrac{da}{dt} - 4t^3 = 6\left(a^2 + 2at\dfrac{da}{dt}\right)$, solve for $\dfrac{da}{dt}$ to get $\dfrac{da}{dt} = \dfrac{2t^3 + 3a^2}{2a^3 - 6at}$

34. $\dfrac{1}{2}u^{-1/2}\dfrac{du}{dv} + \dfrac{1}{2}v^{-1/2} = 0$ so $\dfrac{du}{dv} = -\dfrac{\sqrt{u}}{\sqrt{v}}$

35. $2a^2\omega\dfrac{d\omega}{d\lambda} + 2b^2\lambda = 0$ so $\dfrac{d\omega}{d\lambda} = -\dfrac{b^2\lambda}{a^2\omega}$

36. $1 = (\cos x)\dfrac{dx}{dy}$ so $\dfrac{dx}{dy} = \dfrac{1}{\cos x} = \sec x$