78. $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dv}\dfrac{dv}{dw}\dfrac{dw}{dx}$

79. $\dfrac{d}{dx}[f(g(h(x)))] = \dfrac{d}{dx}[f(g(u))], \quad u = h(x)$

$$= \frac{d}{du}[f(g(u))]\frac{du}{dx} = f'(g(u))g'(u)\frac{du}{dx} = f'(g(h(x)))g'(h(x))h'(x)$$

EXERCISE SET 3.7

1. $\dfrac{dy}{dt} = 3\dfrac{dx}{dt}$

(a) $\dfrac{dy}{dt} = 3(2) = 6$

(b) $-1 = 3\dfrac{dx}{dt}, \ \dfrac{dx}{dt} = -\dfrac{1}{3}$

2. $\dfrac{dx}{dt} + 4\dfrac{dy}{dt} = 0$

(a) $1 + 4\dfrac{dy}{dt} = 0$ so $\dfrac{dy}{dt} = -\dfrac{1}{4}$ when $x = 2$.

(b) $\dfrac{dx}{dt} + 4(4) = 0$ so $\dfrac{dx}{dt} = -16$ when $x = 3$.

3. $8x\dfrac{dx}{dt} + 18y\dfrac{dy}{dt} = 0$

(a) $8\dfrac{1}{2\sqrt{2}} \cdot 3 + 18\dfrac{1}{3\sqrt{2}}\dfrac{dy}{dt} = 0, \ \dfrac{dy}{dt} = -2$

(b) $8\left(\dfrac{1}{3}\right)\dfrac{dx}{dt} - 18\dfrac{\sqrt{5}}{9} \cdot 8 = 0, \ \dfrac{dx}{dt} = 6\sqrt{5}$

4. $2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 2\dfrac{dx}{dt} + 4\dfrac{dy}{dt}$

(a) $2 \cdot 3(-5) + 2 \cdot 1\dfrac{dy}{dt} = 2(-5) + 4\dfrac{dy}{dt}, \ \dfrac{dy}{dt} = -10$

(b) $2(1+\sqrt{2})\dfrac{dx}{dt} + 2(2+\sqrt{3}) \cdot 6 = 2\dfrac{dx}{dt} + 4 \cdot 6, \ \dfrac{dx}{dt} = -12\dfrac{\sqrt{3}}{2\sqrt{2}} = -3\sqrt{3}\sqrt{2}$

5. (b) $A = x^2$

(c) $\dfrac{dA}{dt} = 2x\dfrac{dx}{dt}$

(d) Find $\left.\dfrac{dA}{dt}\right|_{x=3}$ given that $\left.\dfrac{dx}{dt}\right|_{x=3} = 2$. From Part (c), $\left.\dfrac{dA}{dt}\right|_{x=3} = 2(3)(2) = 12 \text{ ft}^2/\text{min}$.

6. (b) $A = \pi r^2$

(c) $\dfrac{dA}{dt} = 2\pi r\dfrac{dr}{dt}$

(d) Find $\left.\dfrac{dA}{dt}\right|_{r=5}$ given that $\left.\dfrac{dr}{dt}\right|_{r=5} = 2$. From Part (c), $\left.\dfrac{dA}{dt}\right|_{r=5} = 2\pi(5)(2) = 20\pi \text{ cm}^2/\text{s}$.

7. (a) $V = \pi r^2 h$, so $\dfrac{dV}{dt} = \pi\left(r^2\dfrac{dh}{dt} + 2rh\dfrac{dr}{dt}\right)$.

(b) Find $\left.\dfrac{dV}{dt}\right|_{\substack{h=6,\\ r=10}}$ given that $\left.\dfrac{dh}{dt}\right|_{\substack{h=6,\\ r=10}} = 1$ and $\left.\dfrac{dr}{dt}\right|_{\substack{h=6,\\ r=10}} = -1$. From Part (a),

$\left.\dfrac{dV}{dt}\right|_{\substack{h=6,\\ r=10}} = \pi[10^2(1) + 2(10)(6)(-1)] = -20\pi \text{ in}^3/\text{s}$; the volume is decreasing.

8. (a) $\ell^2 = x^2 + y^2$, so $\dfrac{d\ell}{dt} = \dfrac{1}{\ell}\left(x\dfrac{dx}{dt} + y\dfrac{dy}{dt}\right)$.

(b) Find $\left.\dfrac{d\ell}{dt}\right|_{\substack{x=3,\\y=4}}$ given that $\dfrac{dx}{dt} = \dfrac{1}{2}$ and $\dfrac{dy}{dt} = -\dfrac{1}{4}$.

From Part (a) and the fact that $\ell = 5$ when $x = 3$ and $y = 4$,

$\left.\dfrac{d\ell}{dt}\right|_{\substack{x=3,\\y=4}} = \dfrac{1}{5}\left[3\left(\dfrac{1}{2}\right) + 4\left(-\dfrac{1}{4}\right)\right] = \dfrac{1}{10}$ ft/s; the diagonal is increasing.

9. (a) $\tan\theta = \dfrac{y}{x}$, so $\sec^2\theta\dfrac{d\theta}{dt} = \dfrac{x\dfrac{dy}{dt} - y\dfrac{dx}{dt}}{x^2}$, $\dfrac{d\theta}{dt} = \dfrac{\cos^2\theta}{x^2}\left(x\dfrac{dy}{dt} - y\dfrac{dx}{dt}\right)$

(b) Find $\left.\dfrac{d\theta}{dt}\right|_{\substack{x=2,\\y=2}}$ given that $\left.\dfrac{dx}{dt}\right|_{\substack{x=2,\\y=2}} = 1$ and $\left.\dfrac{dy}{dt}\right|_{\substack{x=2,\\y=2}} = -\dfrac{1}{4}$.

When $x = 2$ and $y = 2$, $\tan\theta = 2/2 = 1$ so $\theta = \dfrac{\pi}{4}$ and $\cos\theta = \cos\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$. Thus from Part (a), $\left.\dfrac{d\theta}{dt}\right|_{\substack{x=2,\\y=2}} = \dfrac{(1/\sqrt{2})^2}{2^2}\left[2\left(-\dfrac{1}{4}\right) - 2(1)\right] = -\dfrac{5}{16}$ rad/s; θ is decreasing.

10. Find $\left.\dfrac{dz}{dt}\right|_{\substack{x=1,\\y=2}}$ given that $\left.\dfrac{dx}{dt}\right|_{\substack{x=1,\\y=2}} = -2$ and $\left.\dfrac{dy}{dt}\right|_{\substack{x=1,\\y=2}} = 3$.

$\dfrac{dz}{dt} = 2x^3y\dfrac{dy}{dt} + 3x^2y^2\dfrac{dx}{dt}$, $\left.\dfrac{dz}{dt}\right|_{\substack{x=1,\\y=2}} = (4)(3) + (12)(-2) = -12$ units/s; z is decreasing.

11. Let A be the area swept out, and θ the angle through which the minute hand has rotated. Find $\dfrac{dA}{dt}$ given that $\dfrac{d\theta}{dt} = \dfrac{\pi}{30}$ rad/min; $A = \dfrac{1}{2}r^2\theta = 8\theta$, so $\dfrac{dA}{dt} = 8\dfrac{d\theta}{dt} = \dfrac{4\pi}{15}$ in^2/min.

12. Let r be the radius and A the area enclosed by the ripple. We want $\left.\dfrac{dA}{dt}\right|_{t=10}$ given that $\dfrac{dr}{dt} = 3$. We know that $A = \pi r^2$, so $\dfrac{dA}{dt} = 2\pi r\dfrac{dr}{dt}$. Because r is increasing at the constant rate of 3 ft/s, it follows that $r = 30$ ft after 10 seconds so $\left.\dfrac{dA}{dt}\right|_{t=10} = 2\pi(30)(3) = 180\pi$ ft^2/s.

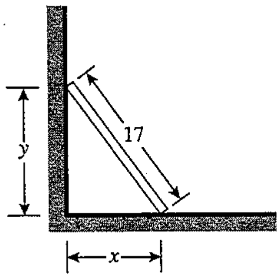
13. Find $\left.\dfrac{dr}{dt}\right|_{A=9}$ given that $\dfrac{dA}{dt} = 6$. From $A = \pi r^2$ we get $\dfrac{dA}{dt} = 2\pi r\dfrac{dr}{dt}$ so $\dfrac{dr}{dt} = \dfrac{1}{2\pi r}\dfrac{dA}{dt}$. If $A = 9$ then $\pi r^2 = 9$, $r = 3/\sqrt{\pi}$ so $\left.\dfrac{dr}{dt}\right|_{A=9} = \dfrac{1}{2\pi(3/\sqrt{\pi})}(6) = 1/\sqrt{\pi}$ mi/h.

14. The volume V of a sphere of radius r is given by $V = \dfrac{4}{3}\pi r^3$ or, because $r = \dfrac{D}{2}$ where D is the diameter, $V = \dfrac{4}{3}\pi\left(\dfrac{D}{2}\right)^3 = \dfrac{1}{6}\pi D^3$. We want $\left.\dfrac{dD}{dt}\right|_{r=1}$ given that $\dfrac{dV}{dt} = 3$. From $V = \dfrac{1}{6}\pi D^3$ we get $\dfrac{dV}{dt} = \dfrac{1}{2}\pi D^2\dfrac{dD}{dt}$, $\dfrac{dD}{dt} = \dfrac{2}{\pi D^2}\dfrac{dV}{dt}$, so $\left.\dfrac{dD}{dt}\right|_{r=1} = \dfrac{2}{\pi(2)^2}(3) = \dfrac{3}{2\pi}$ ft/min.

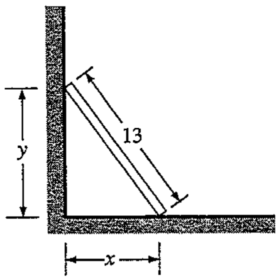
15. Find $\left.\frac{dV}{dt}\right|_{r=9}$ given that $\frac{dr}{dt} = -15$. From $V = \frac{4}{3}\pi r^3$ we get $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$ so

$\left.\frac{dV}{dt}\right|_{r=9} = 4\pi(9)^2(-15) = -4860\pi$. Air must be removed at the rate of 4860π cm^3/min.

16. Let x and y be the distances shown in the diagram. We want to find $\left.\frac{dy}{dt}\right|_{y=8}$ given that $\frac{dx}{dt} = 5$. From $x^2+y^2 = 17^2$ we get $2x\frac{dx}{dt}+2y\frac{dy}{dt} = 0$, so $\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$. When $y = 8$, $x^2+8^2 = 17^2$, $x^2 = 289-64 = 225$, $x = 15$ so $\left.\frac{dy}{dt}\right|_{y=8} = -\frac{15}{8}(5) = -\frac{75}{8}$ ft/s; the top of the ladder is moving down the wall at a rate of 75/8 ft/s.



17. Find $\left.\frac{dx}{dt}\right|_{y=5}$ given that $\frac{dy}{dt} = -2$. From $x^2 + y^2 = 13^2$ we get $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$ so $\frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt}$. Use $x^2 + y^2 = 169$ to find that $x = 12$ when $y = 5$ so $\left.\frac{dx}{dt}\right|_{y=5} = -\frac{5}{12}(-2) = \frac{5}{6}$ ft/s.



18. Let θ be the acute angle, and x the distance of the bottom of the plank from the wall. Find $\left.\frac{d\theta}{dt}\right|_{x=2}$ given that $\left.\frac{dx}{dt}\right|_{x=2} = -\frac{1}{2}$ ft/s. The variables θ and x are related by the equation $\cos\theta = \frac{x}{10}$ so $-\sin\theta\frac{d\theta}{dt} = \frac{1}{10}\frac{dx}{dt}$, $\frac{d\theta}{dt} = -\frac{1}{10\sin\theta}\frac{dx}{dt}$. When $x = 2$, the top of the plank is $\sqrt{10^2 - 2^2} = \sqrt{96}$ ft above the ground so $\sin\theta = \sqrt{96}/10$ and $\left.\frac{d\theta}{dt}\right|_{x=2} = -\frac{1}{\sqrt{96}}\left(-\frac{1}{2}\right) = \frac{1}{2\sqrt{96}} \approx 0.051$ rad/s.

19. Let x denote the distance from first base and y the distance from home plate. Then $x^2 + 60^2 = y^2$ and $2x\frac{dx}{dt} = 2y\frac{dy}{dt}$. When $x = 50$ then $y = 10\sqrt{61}$ so $\frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt} = \frac{50}{10\sqrt{61}}(25) = \frac{125}{\sqrt{61}}$ ft/s.